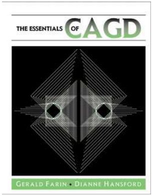
The Essentials of CAGD

Chapter 9: Composite Curves

Gerald Farin & Dianne Hansford

CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/essentials-cagd



Outline

Introduction to Composite Curves

Bézier curves are a powerful tool

One curve not suitable for modeling complex shape

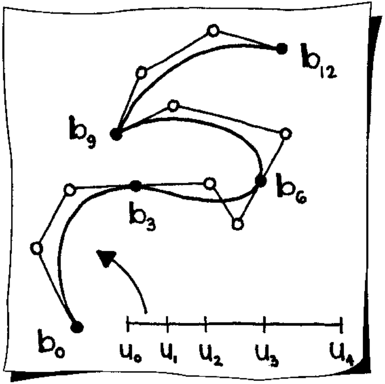
Composite curves:
composed of pieces

Also called
piecewise curves or splines

Examine *piecewise Bézier curves*

– Conditions for smoothness

Piecewise Bézier Curves



knot sequence $u_0, u_1, \ldots$

Each Bézier curve is defined over an interval $[u_i, u_{i+1}]$

$\Delta_i = u_{i+1} - u_i$

spline curve: piecewise curve defined over a knot sequence

Piecewise Bézier Curves

Spline curve $\mathbf{s}(u)$

u: Global parameter within the knot vector

i^{th} Bézier curve $\mathbf{s}_i$
– Defined over $[u_i, u_{i+1}]$
– Local parameter $t \in [0, 1]$

$$t = \frac{u - u_i}{\Delta_i}$$

Junction point: curve segment end points:

$$\mathbf{s}(u_i) = \mathbf{s}_i(0) = \mathbf{s}_{i-1}(1)$$

Piecewise Bézier Curves

Derivative of a spline curve at u when $u \in [u_i, u_{i+1}]$

$$\frac{\mathrm{d}\mathbf{s}(u)}{\mathrm{d}u} = \frac{\mathrm{d}\mathbf{s}_i(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{\Delta_i}\frac{\mathrm{d}\mathbf{s}_i(t)}{\mathrm{d}t}$$

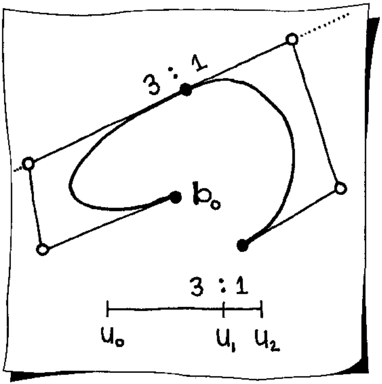
At *junction points* of Bézier curves

$$\frac{1}{\Delta_0}\dot{\mathbf{s}}_0(1) = \frac{3}{\Delta_0}\Delta\mathbf{b}_2 \quad \text{and} \quad \frac{1}{\Delta_1}\dot{\mathbf{s}}_1(0) = \frac{3}{\Delta_1}\Delta\mathbf{b}_3$$

Second derivatives follow similarly:

$$\frac{1}{\Delta_0^2}\ddot{\mathbf{s}}_0(1) = \frac{6}{\Delta_0^2}\Delta^2\mathbf{b}_1 \quad \text{and} \quad \frac{1}{\Delta_1^2}\ddot{\mathbf{s}}_1(0) = \frac{6}{\Delta_1^2}\Delta^2\mathbf{b}_3$$

C^1 and G^1 Continuity



(Note sketch error: not in ratio 3:1)

Conditions for two segments to form a differentiable or C^1 curve over the interval $[u_0, u_2]$:

$$\mathbf{b}_3 = \frac{\Delta_1}{\Delta}\mathbf{b}_2 + \frac{\Delta_0}{\Delta}\mathbf{b}_4$$

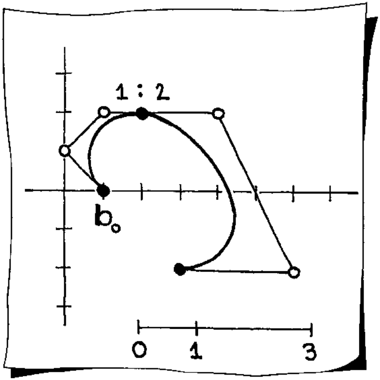
where $\Delta = \Delta_0 + \Delta_1 = u_2 - u_0$

Geometric interpretation:

$$\mathrm{ratio}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4) = \frac{\Delta_0}{\Delta_1}$$

C^1 and G^1 Continuity

Example:



C^1 condition requires

$$\mathbf{b}_3 = \frac{2}{3}\mathbf{b}_2 + \frac{1}{3}\mathbf{b}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

C^1 and G^1 Continuity

Interpret parameter interval $[u_0, u_2]$ as a time interval

C^1 motion $\Rightarrow$ point's velocity must change continuously

Point must travel faster over "long" parameter intervals and slower over "short" ones

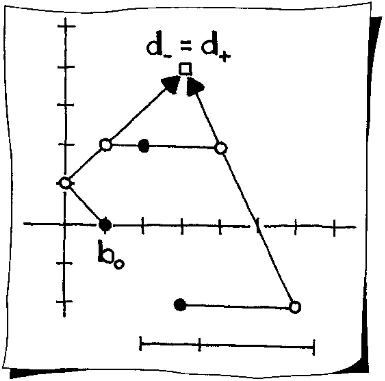
Shape only concern? Then knot sequence not needed

G^1 continuity:

Tangent line varies continuously

– Example: two cubic Bézier curves: $\mathbf{b}_2, \mathbf{b}_3$, and $\mathbf{b}_4$ collinear

C^2 and G^2 Continuity



Assume C^1

Compare second derivatives at parameter value u_1

$$-\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2 = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5$$

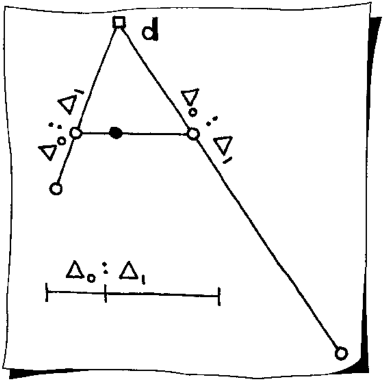
Geometric interpretation:

$$\mathbf{d}_- = -\frac{\Delta_1}{\Delta_0}\mathbf{b}_1 + \frac{\Delta}{\Delta_0}\mathbf{b}_2$$

$$\mathbf{d}_+ = \frac{\Delta}{\Delta_1}\mathbf{b}_4 - \frac{\Delta_0}{\Delta_1}\mathbf{b}_5$$

C^2 condition: $\mathbf{d}_- = \mathbf{d}_+ \equiv \mathbf{d}$

C^2 and G^2 Continuity



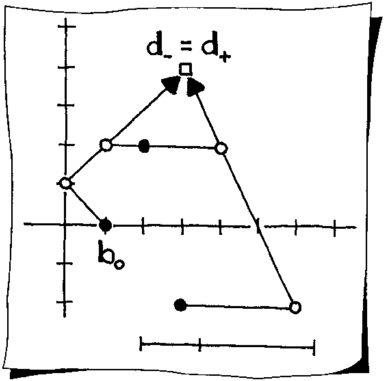
C^2 curves:

$$\mathbf{b}_2 = \frac{\Delta_1}{\Delta}\mathbf{b}_1 + \frac{\Delta_0}{\Delta}\mathbf{d}$$

$$\mathbf{b}_4 = \frac{\Delta_1}{\Delta}\mathbf{d} + \frac{\Delta_0}{\Delta}\mathbf{b}_5$$

$$\mathrm{ratio}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{d}) = \mathrm{ratio}(\mathbf{d}, \mathbf{b}_4, \mathbf{b}_5) = \frac{\Delta_0}{\Delta_1}$$

C^2 and G^2 Continuity



G^2 continuity $\Rightarrow$ *curvature* continuity

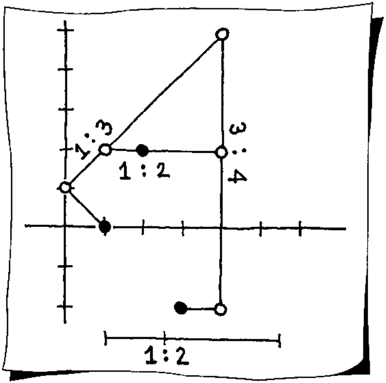
$$\rho^2 = \rho_0 \rho_1$$

where

$$\rho_0 = \mathrm{ratio}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c})$$
$$\rho_1 = \mathrm{ratio}(\mathbf{c}, \mathbf{b}_4, \mathbf{b}_5)$$
$$\rho = \mathrm{ratio}(\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4)$$

C^2 and G^2 Continuity

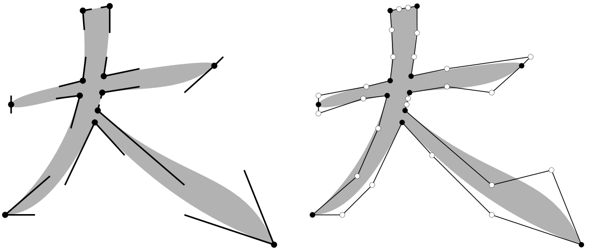
A curve that is G^2 but not C^2



$$\rho_0 = \frac{1}{3} \quad \rho_1 = \frac{3}{4} \quad \rho = \frac{1}{2}$$

$$(\frac{1}{2})^2 = \frac{1}{3} \times \frac{3}{4}$$

Working with Piecewise Bézier Curves



Left: Points and tangent lines   Right: Piecewise Bézier polygon

Given:

– Points that correspond to significant changes in geometry

– Tangent lines at points where the character is smooth

Find: Piecewise Bézier representation

– One tangent line ⇒ Use G^1 smoothness conditions for Bézier curves

Working with Piecewise Bézier Curves

1. Order the points as $\mathbf{p}_i$
2. Convert tangent *lines* to *unit vectors* $\mathbf{v}_i$
3. For one cubic segment set

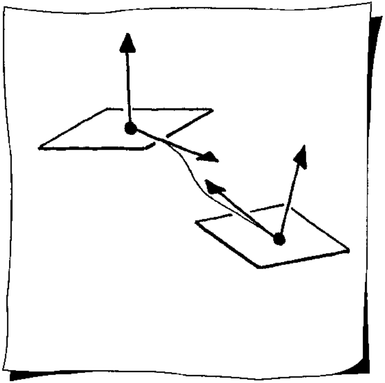
$$\mathbf{b}_{3i} = \mathbf{p}_i \quad \text{and} \quad \mathbf{b}_{3i+3} = \mathbf{p}_{i+1}$$

4. Interior control points

$$\mathbf{b}_{3i+1} = \mathbf{b}_{3i} + 0.4\|\mathbf{b}_{3i+3} - \mathbf{b}_{3i}\|\mathbf{v}_i$$
$$\mathbf{b}_{3i+2} = \mathbf{b}_{3i+3} - 0.4\|\mathbf{b}_{3i+3} - \mathbf{b}_{3i}\|\mathbf{v}_{i+1}$$

Characters or fonts often stored as piecewise Bézier curves
– Allows for easy rescaling
– Pixel maps of fonts can result in aliasing effects
– Each letter in this book is created by evaluating a piecewise Bézier curve

Point-Normal Interpolation



Given:

– A pair of 3D points $\mathbf{p}_0, \mathbf{p}_1$

– Normal vectors $\mathbf{n}_0, \mathbf{n}_1$ at each $\mathbf{p}_i$

Find:

– Cubic connecting $\mathbf{p}_0$ and $\mathbf{p}_1$

– Curve is tangent to the planes defined normal vectors

$\Rightarrow$ Curve's tangents lie planes

Point-Normal Interpolation

In Bézier form:

$\mathbf{b}_0 = \mathbf{p}_0$ and $\mathbf{b}_3 = \mathbf{p}_1$

Infinitely many solutions for $\mathbf{b}_1$ and $\mathbf{b}_2$

One solution:

1. Project $\mathbf{b}_3$ into the plane defined by $\mathbf{b}_0$ and $\mathbf{n}_0$
   This defines a tangent line at $\mathbf{b}_0$
2. Place $\mathbf{b}_1$ anywhere on this tangent
   Could use method in previous section
3. $\mathbf{b}_2$ obtained analogously

Application: robotics

– Path of a robot arm described as a piecewise curve

– Point-normal pairs extracted from a surface

– Desired curve intended to lie on the surface