

Practical Linear Algebra  
A Geometry Toolbox  
Third Edition

Gerald Farin  
Dianne Hansford



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# Preface

Just about everyone has watched animated movies, such as *Toy Story* or *Shrek*, or is familiar with the latest three-dimensional computer games. Enjoying 3D entertainment sounds like more fun than studying a linear algebra book. But it is because of linear algebra that those movies and games can be brought to a TV or computer screen. When you see a character move on the screen, it's animated using some equation straight out of this book. In this sense, linear algebra is a driving force of our new digital world: it is powering the software behind modern visual entertainment and communication.

But this is not a book on entertainment. We start with the fundamentals of linear algebra and proceed to various applications. So it doesn't become too dry, we replaced mathematical proofs with motivations, examples, or graphics. For a beginning student, this will result in a deeper level of understanding than standard theorem-proof approaches. The book covers all of undergraduate-level linear algebra in the classical sense—except it is not delivered in a classical way. Since it relies heavily on examples and pointers to applications, we chose the title *Practical Linear Algebra*, or *PLA* for short.

The subtitle of this book is *A Geometry Toolbox*; this is meant to emphasize that we approach linear algebra in a geometric and algorithmic way. Our goal is to bring the material of this book to a broader audience, motivated in a large part by our observations of how little engineers and scientists (non-math majors) retain from classical linear algebra classes. Thus, we set out to fill a void in the linear algebra textbook market. We feel that we have achieved this, presenting the material in an intuitive, geometric manner that will lend itself to retention of the ideas and methods.

## Review of Contents

As stated previously, one clear motivation we had for writing PLA was to present the material so that the reader would retain the information. In our experience, approaching the material first in two and then in three dimensions lends itself to visualizing and then to understanding. Incorporating many illustrations, Chapters 1–7 introduce the fundamentals of linear algebra in a 2D setting. These same concepts are revisited in Chapters 8–11 in a 3D setting. The 3D world lends itself to concepts that do not exist in 2D, and these are explored there too.

Higher dimensions, necessary for many real-life applications and the development of abstract thought, are visited in Chapters 12–16. The focus of these chapters includes linear system solvers (Gauss elimination, LU decomposition, the Householder method, and iterative methods), determinants, inverse matrices, revisiting “eigen things,” linear spaces, inner products, and the Gram-Schmidt process. Singular value decomposition, the pseudoinverse, and principal components analysis are new additions.

Conics, discussed in Chapter 19, are a fundamental geometric entity, and since their development provides a wonderful application for affine maps, “eigen things,” and symmetric matrices, they really shouldn’t be missed. Triangles in Chapter 17 and polygons in Chapter 18 are discussed because they are fundamental geometric entities and are important in generating computer images.

Several of the chapters have an “Application” section, giving a real-world use of the tools developed thus far. We have made an effort to choose applications that many readers will enjoy by staying away from in-depth domain-specific language. Chapter 20 may be viewed as an application chapter as a whole. Various linear algebra ingredients are applied to the techniques of curve design and analysis.

The illustrations in the book come in two forms: figures and sketches. The figures are computer generated and tend to be complex. The sketches are hand-drawn and illustrate the core of a concept. Both are great teaching and learning tools! We made all of them available on the book’s website <http://www.farinahansford.com/books/pla/>. Many of the figures were generated using PostScript, an easy-to-use geometric language, or Mathematica.

At the end of each chapter, we have included a list of topics, *What You Should Know* (WYSK), marked by the icon on the left. This list is intended to encapsulate the main points of each chapter. It is not uncommon for a topic to appear in more than one chapter. We have



made an effort to revisit some key ideas more than once. Repetition is useful for retention!

Exercises are listed at the end of each chapter. Solutions to selected exercises are given in Appendix B. All solutions are available to instructors and instructions for accessing these may be found on the book's website.

Appendix A provides an extensive glossary that can serve as a review tool. We give brief definitions without equations so as to present a different presentation than that in the text. Also notable is the robust index, which we hope will be very helpful, particularly since we revisit topics throughout the text.

## Classroom Use

PLA is meant to be used at the undergraduate level. It serves as an introduction to linear algebra for engineers or computer scientists, as well as a general introduction to geometry. It is also an ideal preparation for computer graphics and geometric modeling. We would argue that it is also a perfect linear algebra entry point for mathematics majors.

As a one-semester course, we recommend choosing a subset of the material that meets the needs of the students. In the table below, LA refers to an introductory linear algebra course and CG refers to a course tailored to those planning to work in computer graphics or geometric modeling.

	<b>Chapter</b>	<b>LA</b>	<b>CG</b>
1	Descartes' Discovery	•	•
2	Here and There: Points and Vectors in 2D	•	•
3	Lining Up: 2D Lines		•
4	Changing Shapes: Linear Maps in 2D	•	•
5	$2 \times 2$ Linear Systems	•	•
6	Moving Things Around: Affine Maps in 2D	•	•
7	Eigen Things	•	
8	3D Geometry	•	•
9	Linear Maps in 3D	•	•
10	Affine Maps in 3D	•	•
11	Interactions in 3D		•

Chapter		LA	CG
12	Gauss for Linear Systems	•	•
13	Alternative System Solvers	•	
14	General Linear Spaces	•	
15	Eigen Things Revisited	•	
16	The Singular Value Decomposition	•	
17	Breaking It Up: Triangles		•
18	Putting Lines Together: Polylines and Polygons		•
19	Conics		•
20	Curves		•

## Website

*Practical Linear Algebra, A Geometry Toolbox* has a website:  
<http://www.farinhansford.com/books/pla/>

This website provides:

- teaching materials,
- additional material,
- the PostScript files illustrated in the book,
- Mathematica code,
- errata,
- and more!

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