Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 1: Descartes' Discovery

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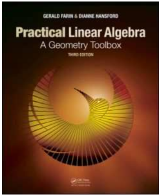
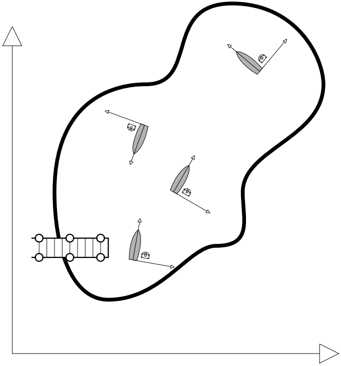
www.farinhansford.com/books/pla



Outline

Introduction to Descartes' Discovery

Tale of Schilda: Save the town treasure! Hide it in the lake. Where? We'll make a notch in the boat to record treasure location. Good idea?



Local and global coordinate systems:

Treasure's local coordinates with respect to the boat do not change as the boat moves.

Treasure's global coordinates with respect to the lake do change as the boat moves.

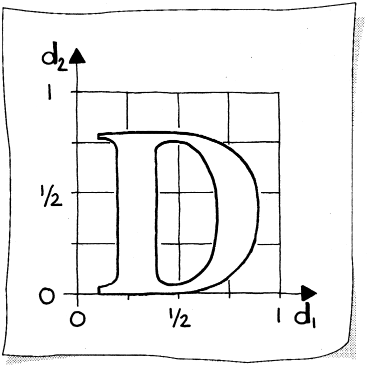
Introduction to Descartes' Discovery

This chapter is about the interplay of local and global coordinates systems

René Descartes (1596-1650)

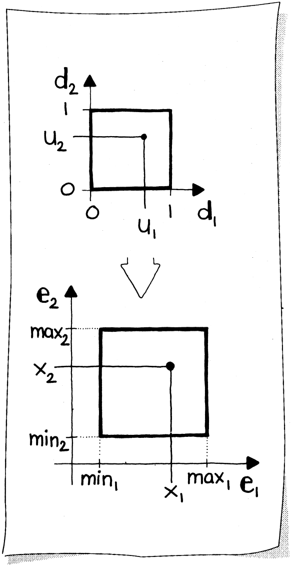
- French philosopher, mathematician, and writer
- Invented the theory of *coordinate systems*
- Cartesian coordinates (Descartes in Latin is Cartesius)
- Key for linking algebra and geometry

Local and Global Coordinates: 2D



Font design example:
a local 2D coordinate system

Local and Global Coordinates: 2D



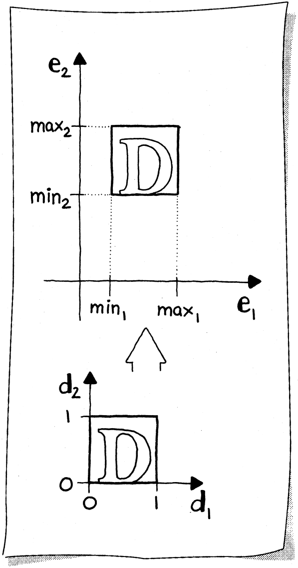
Top: local system
Bottom: global system

Map local (u_1, u_2) to global (x_1, x_2)

$$x_1 = (1 - u_1)\mathrm{min}_1 + u_1\mathrm{max}_1$$
$$x_2 = (1 - u_2)\mathrm{min}_2 + u_2\mathrm{max}_2$$

Local coordinates also called *parameters*

Local and Global Coordinates: 2D



Local and global **D**

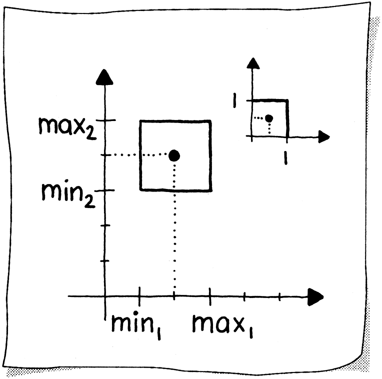
Check mapping of $(u_1, u_2) = (0, 0)$:

$$x_1 = (1 - 0) \cdot \mathrm{min}_1 + 0 \cdot \mathrm{max}_1 = \mathrm{min}_1$$
$$x_2 = (1 - 0) \cdot \mathrm{min}_2 + 0 \cdot \mathrm{max}_2 = \mathrm{min}_2$$

Check mapping of $(u_1, u_2) = (1, 0)$:

$$x_1 = (1 - 1) \cdot \mathrm{min}_1 + 1 \cdot \mathrm{max}_1 = \mathrm{max}_1$$
$$x_2 = (1 - 0) \cdot \mathrm{min}_2 + 0 \cdot \mathrm{max}_2 = \mathrm{min}_2$$

Local and Global Coordinates: 2D



Example:

Given target box

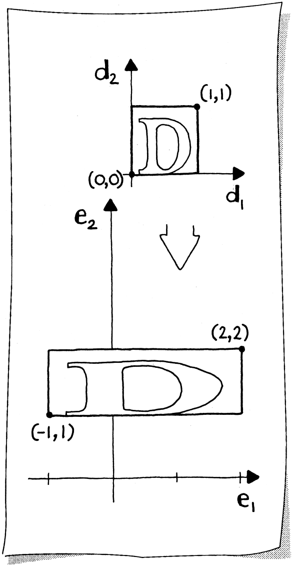
$$(\min_1, \min_2) = (1, 3)$$
$$(\max_1, \max_2) = (3, 5)$$

Local midpoint $(1/2, 1/2)$ maps to

$$x_1 = (1 - \frac{1}{2}) \cdot 1 + \frac{1}{2} \cdot 3 = 2$$
$$x_2 = (1 - \frac{1}{2}) \cdot 3 + \frac{1}{2} \cdot 5 = 4$$

midpoint of the target box

Local and Global Coordinates: 2D



Target box need not be square:

$$(\min_1, \min_2) = (-1, 1)$$
$$(\max_1, \max_2) = (2, 2)$$

Results in a distortion of **D**

Local to global transformation:

$$x_1 = (1 - u_1)\mathrm{min}_1 + u_1\mathrm{max}_1$$
$$x_2 = (1 - u_2)\mathrm{min}_2 + u_2\mathrm{max}_2$$

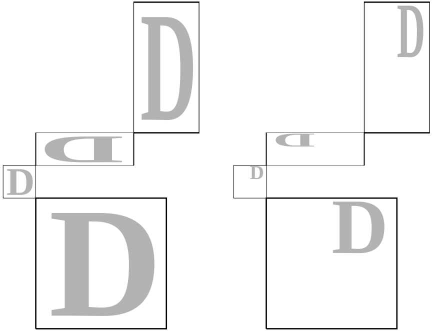
Define $\Delta_1 = \mathrm{max}_1 - \mathrm{min}_1$ and $\Delta_2 = \mathrm{max}_2 - \mathrm{min}_2$.
Now we have

$$x_1 = \mathrm{min}_1 + u_1\Delta_1$$
$$x_2 = \mathrm{min}_2 + u_2\Delta_2$$

Aspect ratio is ratio of width to height: Δ_1/Δ_2 or $\Delta_1 : \Delta_2$

- Old television: $4 : 3$ (nearly square)
- New television: $16 : 9$
- International (ISO A series) paper: $1 : \sqrt{2}$

Local and Global Coordinates: 2D



Target boxes:

Letter **D** mapped several times.

Left: centered in the unit square.

Right: not centered

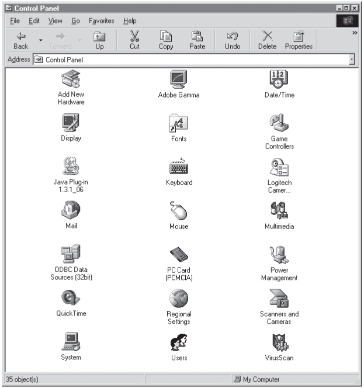
Δ_1 and geometry in $\mathbf{e}_1$-direction:

- $\Delta_1 > 1$: stretch
- $0 < \Delta_1 < 1$: shrink
- $\Delta_1 < 0$: reverse

Same idea for

Δ_2 and geometry in $\mathbf{e}_2$-direction

Example: selecting an icon



Pixel extents of window on screen:

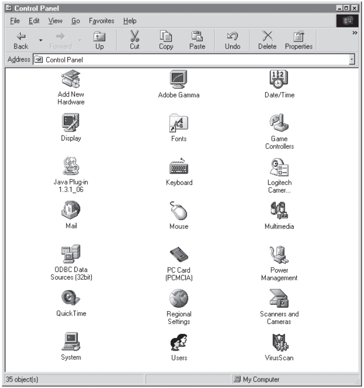
$$(\min_1, \min_2) = (120, 300)$$
$$(\max_1, \max_2) = (600, 820)$$

7×3 icons live in
local coordinate partition:

$$0, 0.33, 0.67, 1$$
$$0, 0.14, 0.29, 0.43, 0.57, 0.71, 0.86, 1$$

Going from Global to Local

$$u_1 = \frac{x_1 - \min_1}{\Delta_1} \qquad u_2 = \frac{x_2 - \min_2}{\Delta_2}$$



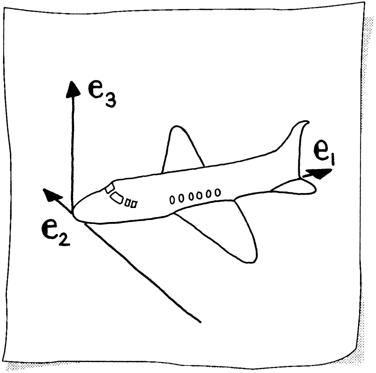
Mouse click returns (200, 709)

$$u_1 = \frac{200 - 120}{480} \approx 0.17$$

$$u_2 = \frac{709 - 300}{520} \approx 0.79$$

Find local coordinates in partition "Display" icon picked

Local and Global Coordinates: 3D



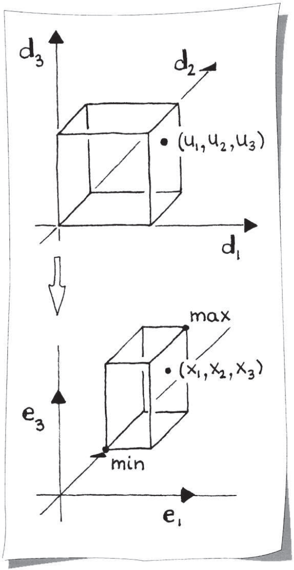
Engineering objects designed using *Computer Aided Design* (*CAD*) system

Every object defined in a (local) coordinate system

Many individual objects integrated into one (global) coordinate system

Example: airplane
engines, seats, wheels, body, etc.

Local and Global Coordinates: 3D



Local 3D coordinate system:
$[\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3]$-system
Coordinates (u_1, u_2, u_3)
Defining *unit cube*

$$0 \le u_1, u_2, u_3 \le 1$$

Cube mapped to *3D target box* in global $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system
Target box extents:

$$(\mathrm{min}_1, \mathrm{min}_2, \mathrm{min}_3)$$

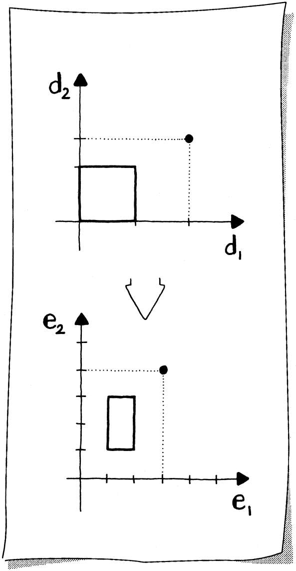
$$(\mathrm{max}_1, \mathrm{max}_2, \mathrm{max}_3)$$

$$x_1 = (1 - u_1)\mathrm{min}_1 + u_1 \mathrm{max}_1$$

$$x_2 = (1 - u_2)\mathrm{min}_2 + u_2 \mathrm{max}_2$$

$$x_3 = (1 - u_3)\mathrm{min}_3 + u_3 \mathrm{max}_3$$

Stepping Outside the Box



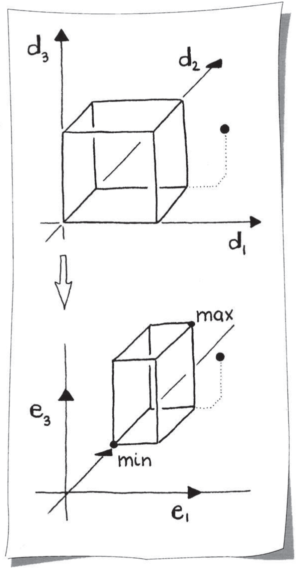
2D coordinate outside the target box
Target box given by

$$(\min_1, \min_2) = (1, 1)$$
$$(\max_1, \max_2) = (2, 3)$$

Coordinates $(u_1, u_2) = (2, 3/2)$
not inside the $[\mathbf{d}_1, \mathbf{d}_2]$-system
Corresponding global coordinates:

$$x_1 = -\min_1 + 2\max_1 = 3,$$
$$x_2 = -\frac{1}{2}\min_2 + \frac{3}{2}\max_2 = 4$$

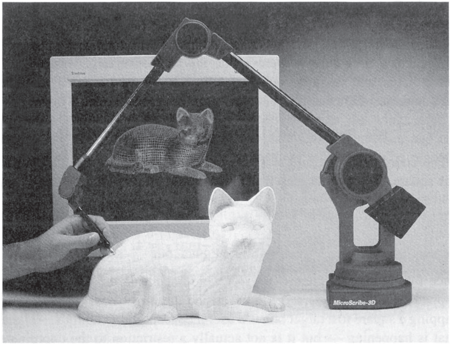
Stepping Outside the Box



3D coordinates outside the target box

Application: Creating Coordinates

Digitizing: Real object ⇒ digital object
Cat “discretized” – turned into a finite number of coordinate triples



Coordinate Measuring Machine (CMM)
Arm records the position of its tip

Touch three points on the table to establish 3D coordinate system

Touch cat model to record coordinates for position

Points called a point cloud

WYSK

- unit square
- 2D and 3D local coordinates
- 2D and 3D global coordinates
- coordinate transformation
- parameter
- aspect ratio
- normalized coordinates
- digitizing
- point cloud