Outline

1. Introduction to Affine Maps in 3D
2. Affine Maps
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6. Homogeneous Coordinates and Perspective Maps
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Affine maps in 3D: fighter jets twisting and turning through 3D space

Affine maps in 3D are a primary tool for modeling and computer graphics.

Additional topic in this chapter: projective maps — the maps used to create realistic 3D images — not affine maps but an important class of maps.
Affine Maps

An affine map in 3D

Linear maps relate vectors to vectors
Affine maps relate points to points
A 3D affine map:

\[ \mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o}) \]

where \( \mathbf{x}, \mathbf{o}, \mathbf{p}, \mathbf{x}' \) are 3D points and \( A \) is a \( 3 \times 3 \) matrix
In general: assume that the origin of \( \mathbf{o} = \mathbf{0} \) and drop it — resulting in

\[ \mathbf{x}' = \mathbf{p} + A\mathbf{x} \]
Affine Maps

Property of affine maps:

Affine maps leave ratios invariant
This map is a rigid body motion
Affine Maps

Property of affine maps:

Affine maps take *parallel planes* to parallel planes
Affine maps take *intersecting planes* to intersecting planes — the intersection line of the mapped planes is the map of the original intersection line
Affine Maps

Property of affine maps:

Affine maps leave *barycentric combinations* invariant.

If

\[ x = c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 \]

where \( c_1 + c_2 + c_3 + c_4 = 1 \)

then after an affine map

\[ x' = c_1 p'_1 + c_2 p'_2 + c_3 p'_3 + c_4 p'_4 \]

Example: the centroid of a tetrahedron \( \Rightarrow \) centroid of the mapped tetrahedron.
A translation is simply an affine map

\[ \mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o}) \]

with \( A = I \) (3 \times 3 identity matrix)

Commonly: \( \mathbf{o} = \mathbf{0} \)

\([\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]\)-system has coordinate axes parallel to \([\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\)-system

Translation is a rigid body motion — Volume of object not changed
Mapping Tetrahedra

A 3D affine map is determined by four point pairs

\[ p_i \rightarrow p'_i \quad \text{for } i = 1, 2, 3, 4 \]

⇒ map determined by a tetrahedron and its image

What is the image of an arbitrary point \( x \) under this affine map?

Key: affine maps leave *barycentric combinations* unchanged

\[
\begin{align*}
x &= u_1 p_1 + u_2 p_2 + u_3 p_3 + u_4 p_4 \\
x' &= u_1 p'_1 + u_2 p'_2 + u_3 p'_3 + u_4 p'_4
\end{align*}
\] (*)

Must find the \( u_i \) — the *barycentric coordinates* of \( x \) with respect to the \( p_i \)

Linear system: three coordinate equations from (*) and one from

\[ u_1 + u_2 + u_3 + u_4 = 1 \Rightarrow \text{four equations for the four unknowns } u_1, \ldots, u_4 \]

(This problem is analogous to 2D case with a triangle)
Mapping Tetrahedra

Example:

Original tetrahedron be given by $p_i$:

$$
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
$$

Map this tetrahedron to points $p'_i$:

$$
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
-1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}
$$

Where is point $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ mapped?
Mapping Tetrahedra

For this simple example we see that

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix} = -2 \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

Barycentric coordinates of \( \mathbf{x} \) with respect to the original \( \mathbf{p}_i \); are \((-2, 1, 1, 1)\)

— Note: they sum to one

\[
\mathbf{x}' = -2 \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
-1 \\
0 \\
-1 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
-1 \\
0 \\
-1
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 \\
-1 \\
-1
\end{bmatrix}
\]
Mapping Tetrahedra

A different approach to constructing the affine map:

Construct a coordinate system from the $p_i$
— Choose $p_1$ as the origin
— Three axes are defined as $p_i - p_1 \; i = 2, 3, 4$

Coordinate system of the $p'_i$ based on the same indices
Find the $3 \times 3$ matrix $A$ and point $p$ that describe the affine map

$$x' = A[x - p_1] + p'_1 \quad \rightarrow \quad p = p'_1$$

We know that:

$$A[p_2 - p_1] = p'_2 - p'_1 \quad A[p_3 - p_1] = p'_3 - p'_1 \quad A[p_4 - p_1] = p'_4 - p'_1$$

Written in matrix form:

$$A \begin{bmatrix} p_2 - p_1 & p_3 - p_1 & p_4 - p_1 \end{bmatrix} = \begin{bmatrix} p'_2 - p'_1 & p'_3 - p'_1 & p'_4 - p'_1 \end{bmatrix}$$

$$A = \begin{bmatrix} p'_2 - p'_1 & p'_3 - p'_1 & p'_4 - p'_1 \end{bmatrix} \begin{bmatrix} p_2 - p_1 & p_3 - p_1 & p_4 - p_1 \end{bmatrix}^{-1}$$
Mapping Tetrahedra

Revisiting the previous example

Select $p_1$ as the origin for the $p_i$ coordinate system
— since $p_1 = 0$ ⇒ no translation

Compute $A$:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\]

\[
x' = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
-1 \\
-1 \\
\end{bmatrix}
\]

Same as barycentric coordinate approach
Parallel Projections

Projections in 3D: a 3D helix is projected into two different 2D planes

Earlier: looked at orthogonal parallel projections as basic linear maps

Everything we draw is a projection of necessity
—paper is 2D after all

Next: projections in the context of 3D affine maps
— maps 3D points onto a plane
Parallel Projections

\( \mathbf{x} \) is projected to \( \mathbf{x}_p \)

Parallel projection: defined by

- direction of projection \( \mathbf{d} \) and
- projection plane \( \mathcal{P} \)

Defines a projection angle \( \theta \)
between \( \mathbf{d} \) and line joining \( \mathbf{x}_o \) in \( \mathcal{P} \)

Angle categorizes parallel projections

- orthogonal or oblique

Orthogonal (orthographic) projections are special

- \( \mathbf{d} \) perpendicular to the plane

Special names for many projection angles
Parallel Projections

Projecting a point on a plane

Project $\mathbf{x}$ in direction $\mathbf{v}$
Projection plane $[\mathbf{x'} - \mathbf{q}] \cdot \mathbf{n} = 0$

Find the projected point $\mathbf{x'} = \mathbf{p} + t\mathbf{v}$
$\Rightarrow$ find $t$

$$[\mathbf{x} + t\mathbf{v} - \mathbf{q}] \cdot \mathbf{n} = 0$$
$$[\mathbf{x} - \mathbf{q}] \cdot \mathbf{n} + t\mathbf{v} \cdot \mathbf{n} = 0$$

$$t = \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}$$

Intersection point $\mathbf{x'}$ computed as

$$\mathbf{x'} = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}$$
Parallel Projections

How to write

\[ x' = x + \frac{[q - x] \cdot n}{v \cdot n} \]

as an affine map in the form \( Ax + p \)?

\[ x' = x - \frac{n \cdot x}{v \cdot n} v + \frac{q \cdot n}{v \cdot n} v \]

Write dot products in matrix form:

\[ x' = x - \frac{n^T x}{v \cdot n} v + \frac{q \cdot n}{v \cdot n} v \]

Observe that \( [n^T x] v = v [n^T x] \)
Matrix multiplication is associative: \( v [n^T x] = [vn^T] x \)
— Notice that \( vn^T \) is a \( 3 \times 3 \) matrix

\[ x' = \left[ I - \frac{vn^T}{v \cdot n} \right] x + \frac{q \cdot n}{v \cdot n} v \]

Achieved form \( x' = Ax + p \)
Parallel Projections

Check the properties of

\[
x' = \left[ I - \frac{vn^T}{v \cdot n} \right] x + \frac{q \cdot n}{v \cdot n} v \quad \text{where } A = I - \frac{vn^T}{v \cdot n}
\]

Projection matrix \( A \) has rank two \( \Rightarrow \) reduces dimensionality

Map is idempotent:

\[
A^2 = (I - \frac{vn^T}{v \cdot n})(I - \frac{vn^T}{v \cdot n})
\]

\[
= I^2 - 2\frac{vn^T}{v \cdot n} + \left(\frac{vn^T}{v \cdot n}\right)^2
\]

\[
= A - \frac{vn^T}{v \cdot n} + \left(\frac{vn^T}{v \cdot n}\right)^2
\]

Expanding the squared term

\[
\frac{vn^Tvn^T}{(v \cdot n)^2} = \frac{vn^T}{v \cdot n}
\]

and thus \( A^2 = A \)
Parallel Projections

Repeating the affine map is idempotent as well:

\[ A(Ax + p) + p = A^2x + Ap + p \]
\[ = Ax + Ap + p \]

Let \( \alpha = (q \cdot n)/(v \cdot n) \) — examine the middle term

\[ Ap = (I - \frac{vn^T}{vn})\alpha v \]
\[ = \alpha v - \alpha v\left(\frac{n^Tv}{vn}\right) \]
\[ = 0 \]

\[ \Rightarrow A(Ax + p) + p = Ax + p \]
\[ \Rightarrow \text{the affine map is idempotent} \]
Parallel Projections

Example:

Given:
projection plane $x_1 + x_2 + x_3 - 1 = 0$

$$x = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$
direction $v = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Project $x$ along $v$ onto the plane: what is $x'$?

Plane’s normal $n = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
Parallel Projections

Choose point \( q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) in the plane

Calculate needed quantities:

\[
\begin{align*}
v \cdot n &= -1 \\
\frac{q \cdot n}{v \cdot n} v &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

\[
v n^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1
\end{bmatrix}
\]

Putting all the pieces together:

\[
x' = \left[ I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}
\]
Homogeneous Coordinates and Perspective Maps

Homogeneous matrix form:
Condense \( x' = A x + p \) into just one matrix multiplication \( x' = M x \)

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  1
\end{bmatrix}
\quad \begin{bmatrix}
  x'_1 \\
  x'_2 \\
  x'_3 \\
  1
\end{bmatrix}
\quad \begin{bmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} & p_1 \\
  a_{2,1} & a_{2,2} & a_{2,3} & p_2 \\
  a_{3,1} & a_{3,2} & a_{3,3} & p_3 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

4D point \( \mathbf{x} \) is the homogeneous form of the affine point \( \mathbf{x} \)

Homogeneous representation of a vector \( \mathbf{v} = [v_1 \ v_2 \ v_3 \ 0]^T \)
\( \Rightarrow \mathbf{v}' = M \mathbf{v} \)
Zero fourth component \( \Rightarrow \) disregard the translation
— translation has no effect on vectors
— vector defined as the difference of two points
Advantage of the homogeneous matrix form:
Condenses information into one matrix
— Implemented in the popular computer graphics Application Programmer’s Interface
— Convenient and efficient to have all information in one data structure

Homogeneous point $x$ to affine counterpart $x$: divide through by $x_4$
⇒ one affine point has infinitely many homogeneous representations

**Example:**

\[
\begin{bmatrix}
1 \\
-1 \\
3
\end{bmatrix} \approx \begin{bmatrix}
10 \\
-10 \\
30
\end{bmatrix} \approx \begin{bmatrix}
-2 \\
2 \\
-6
\end{bmatrix}
\]

(Symbol $\approx$ should be read “corresponds to.”)
Revisit a projection problem:
Given point $x$, projection direction $v$, and projection plane $[x' - q] \cdot n = 0$
The projected point

$$x' = \left[ I - \frac{vn^T}{v \cdot n} \right] x + \frac{q \cdot n}{v \cdot n} v$$

The homogeneous matrix form:

$$\begin{bmatrix}
  v \cdot n & 0 & 0 \\
  0 & v \cdot n & 0 \\
  0 & 0 & v \cdot n \\
\end{bmatrix} - vn^T = (q \cdot n)v$$

$$0 \hspace{1cm} 0 \hspace{1cm} 0 \hspace{1cm} v \cdot n$$
Homogeneous Coordinates and Perspective Maps

Perspective projection

Instead of a constant direction \( \mathbf{v} \), perspective projection direction depends on the point \( \mathbf{x} \) — the line from \( \mathbf{x} \) to the origin: \( \mathbf{v} = -\mathbf{x} \)

\[
\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{x} \cdot \mathbf{n}} \mathbf{x} = \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{x} \cdot \mathbf{n}} \mathbf{x}
\]

Homogeneous matrix form:

\[
\mathbf{M} = \begin{bmatrix}
1 & [\mathbf{q} \cdot \mathbf{n}] & \mathbf{o} \\
0 & 0 & 0 \\
0 & 0 & \mathbf{x} \cdot \mathbf{n}
\end{bmatrix}
\]
Homogeneous Coordinates and Perspective Maps

Perspective projections are not affine maps

**Example:** Plane \( x_3 = 1 \) and point on the plane \( q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \)

\( q \cdot n = 1 \) and \( x \cdot n = x_3 \) — resulting in the map \( x' = \frac{1}{x_3} x \)

Take the three points (see previous Sketch)

\[
x_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}
\]

Note: \( x_2 \) is the midpoint of \( x_1 \) and \( x_3 \)  
Their images are

\[
x_1' = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \quad x_2' = \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix} \quad x_3' = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}
\]

The perspective map destroyed the midpoint relation

\[
x_2' = \frac{2}{3} x_1' + \frac{1}{3} x_3'
\]
Homogeneous Coordinates and Perspective Maps

Perspective maps
— do not preserve the ratio of three points
— two parallel lines will not be mapped to parallel lines
— good model for how we perceive 3D space around us

Left: Parallel projection  Right: Perspective projection
Homogeneous Coordinates and Perspective Maps

Experiment by A. Dürer

Study of perspective goes back to the 14th century
Earlier times: artists could not draw realistic 3D images
affine map
translation
affine map properties
barycentric combination
invariant ratios
barycentric coordinates
centroid
mapping four points to four points
parallel projection
orthogonal projection
oblique projection
line and plane intersection
idempotent
dyadic matrix
homogeneous coordinates
perspective projection
rank