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Introduction to Interactions in 3D

Ray tracing: 3D intersections key for rendering a raytraced image

Points, lines, and planes: basic 3D geometry building blocks

Build real objects
⇒ compute with these building blocks
— Example: intersection

(Description of the ray tracing technique is in this chapter)
Distance Between a Point and a Plane

Given:

- Plane \( \mathbf{n} \cdot \mathbf{x} + c = 0 \)
- Point \( \mathbf{p} \)

What is \( \mathbf{p} \)'s distance \( d \) to the plane?

What is \( \mathbf{p} \)'s closest point \( \mathbf{q} \) on the plane?

Similar to the *foot of a point* from Chapter 3 2D Lines
Distance Between a Point and a Plane

Vector \( \mathbf{p} - \mathbf{q} \) must be perpendicular to the plane

\[ \Rightarrow \text{parallel to the plane's normal } \mathbf{n} \]

\[ \mathbf{p} = \mathbf{q} + t\mathbf{n}; \]

Goal: find \( t \)

\( \mathbf{q} \) satisfies the plane equation:

\[ \mathbf{n} \cdot [\mathbf{p} - t\mathbf{n}] + c = 0 \]

\[ t = \frac{c + \mathbf{n} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{n}} \]

\( t = 0 \Rightarrow \mathbf{p} \text{ is on the plane} \)
**Distance Between a Point and a Plane**

**Example:** point and a plane

Plane

\[ x_1 + x_2 + x_3 - 1 = 0 \]

and the point

\[
\mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}
\]

\[
t = \frac{c + \mathbf{n} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{n}} = 2
\]

\[
\mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 2 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
Distance Between a Point and a Plane

Distance of \( p \) to the plane:
\[
d = \|p - q\| = \|tn\| = t\|n\|
\]
If \( n \) normalized: \( \|p - q\| = t \) and
\[
d = c + n \cdot p
\]
If \( t > 0 \) then \( n \) points towards \( p \)
If \( t < 0 \) then \( n \) points away from \( p \)
If a point is very close to a plane can be numerically hard to decide which side it is on
Distance Between Two Lines

Two 3D lines typically do not meet — such lines are called skew. What is the distance between the lines?

\[ l_1 : \mathbf{x}_1(s_1) = \mathbf{p}_1 + s_1 \mathbf{v}_1 \quad l_2 : \mathbf{x}_2(s_2) = \mathbf{p}_2 + s_2 \mathbf{v}_2 \]

\( \mathbf{x}_1 \): the point on \( l_1 \) closest to \( l_2 \)
\( \mathbf{x}_2 \): the point on \( l_2 \) closest to \( l_1 \)

Vector \( \mathbf{x}_2 - \mathbf{x}_1 \) is perpendicular to both \( l_1 \) and \( l_2 \):

\[
[\mathbf{x}_2 - \mathbf{x}_1] \mathbf{v}_1 = 0
\]
\[
[\mathbf{x}_2 - \mathbf{x}_1] \mathbf{v}_2 = 0
\]

or

\[
[\mathbf{p}_2 - \mathbf{p}_1] \mathbf{v}_1 = s_1 \mathbf{v}_1 \cdot \mathbf{v}_1 - s_2 \mathbf{v}_1 \cdot \mathbf{v}_2
\]
\[
[\mathbf{p}_2 - \mathbf{p}_1] \mathbf{v}_2 = s_1 \mathbf{v}_1 \cdot \mathbf{v}_2 - s_2 \mathbf{v}_2 \cdot \mathbf{v}_2
\]

Two equations in the two unknowns \( s_1 \) and \( s_2 \).
Distance Between Two Lines

**Example:** distance between two lines

\[ l_1 : x_1(s_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ l_2 : x_2(s_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]

Linear system

\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Solution \( s_1 = 0 \) and \( s_2 = -1 \) \( \Rightarrow \)

\[ x_1(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2(-1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]
Two 3D lines intersect if $\mathbf{x}_1 = \mathbf{x}_2$

Floating point calculations $\Rightarrow$ round-off error

$\Rightarrow$ accept closeness within a tolerance: $\|\mathbf{x}_1 - \mathbf{x}_2\|^2 < \text{tolerance}$

A condition for two 3D lines to intersect:

$v_1, v_2, p_2 - p_1$ must be coplanar or linearly dependent

$$\det[v_1, v_2, p_2 - p_1] = 0$$

Numerical viewpoint: safer to compare the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$

— Distance tolerance easier to prescribed than volume tolerance
Ray tracing: basic techniques in computer graphics for creating an image

Scene given as an assembly of planes

Lighting computed by tracing light rays through the scene

Ray intersects a plane, it is reflected, then it intersects the next plane, etc.
Given:
— Plane \( P \) (point \( q \) and normal \( n \))
— Line \( l \) (point \( p \) and vector \( v \))

What is their \textit{intersection point} \( x \)?

On plane: \( [x - q] \cdot n = 0 \)

On line: \( x = p + tv \)

Find \( t \)

\[
[p + tv - q] \cdot n = 0
\]

\[
[p - q] \cdot n + tv \cdot n = 0
\]

\[
t = \frac{[q - p] \cdot n}{v \cdot n}
\]

\[
x = p + \frac{[q - p] \cdot n}{v \cdot n}v
\]

Caution: \( v \cdot n \) might be small or zero
— Geometric interpretation?
**Example:** Intersecting a line and a plane

Plane: \( x_1 + x_2 + x_3 - 1 = 0 \)

Line:

\[
p(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Need a point \( q \) on the plane:

Set \( x_1 = x_2 = 0 \) and solve for \( x_3 \)

— resulting in \( x_3 = 1 \)

\[
q - p \cdot n
\]

\[
v \cdot n
\]

\[
t = \frac{[q - p] \cdot n}{v \cdot n} = -3
\]

Intersection point:

\[
x = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
\]
Intersecting a plane with a line when given plane is in *parametric form*.

Unknown intersection point $\mathbf{x}$ must satisfy

$$\mathbf{x} = \mathbf{q} + u_1\mathbf{r}_1 + u_2\mathbf{r}_2$$

$\mathbf{x}$ is also on the line $\mathbf{l}$:

$$\mathbf{p} + t\mathbf{v} = \mathbf{q} + u_1\mathbf{r}_1 + u_2\mathbf{r}_2$$

Three equations in three unknowns

$$\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & -\mathbf{v} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ t \end{bmatrix} = \begin{bmatrix} \mathbf{p} - \mathbf{q} \end{bmatrix}$$
Ray and 3D triangle intersection:
record intersection interior to triangle

Triangle: \( p_1, p_2, p_3 \)  
Ray: \( p, v \)

Find intersection \( \Rightarrow \) find \( t \) to satisfy

\[
p + tv = p_1 + u_1(p_2 - p_1) + u_2(p_3 - p_1)
\]

Linear system:
3 equations, 3 unknowns \( t, u_1, u_2 \)

Intersection point
\[
= u_1p_2 + u_2p_3 + (1 - u_1 - u_2)p_1
\]
— *barycentric coordinates*

Intersection inside the triangle if

\[
0 \leq u_1, u_2 \leq 1
\]
\[
u_1 + u_2 \leq 1
\]
Reflections

Given: point \( x \) on a plane \( P \) and an “incoming” direction \( v \)

What is the reflected or “outgoing” direction \( v' \)?

(Assume \( v, v', n \) unit length)

\( n \) is the angle bisector of \( v \) and \( v' \)

\[-v \cdot n = v' \cdot n\]

Symmetry property: \( cn = v' - v \)

\[-v \cdot n = [cn + v] \cdot n\]

Solve for \( c = -2v \cdot n \)

\[v' = v - [2v \cdot n]n\]
Reflections

\[ v' = v - [2v \cdot n]n \]
\[ = v - 2[v^T n]n \]
\[ = v - 2[nn^T]v \]

\( nn^T \) is a projection matrix
- orthogonal
- symmetric
- dyadic (rank one)

Reflection as a linear map: \( v' = Hv \)

\[ H = I - 2nn^T \]

Householder matrix \( H \)

Chapter 13 — The Householder Method
Given: three planes

\[ \mathbf{n}_1 \cdot \mathbf{x} + c_1 = 0 \]
\[ \mathbf{n}_2 \cdot \mathbf{x} + c_2 = 0 \]
\[ \mathbf{n}_3 \cdot \mathbf{x} + c_3 = 0 \]

Where do they intersect?

Plane equations into matrix form:

\[
\begin{bmatrix}
\mathbf{n}_1^T \\
\mathbf{n}_2^T \\
\mathbf{n}_3^T
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-c_1 \\
-c_2 \\
-c_3
\end{bmatrix}
\]

Solve three equations in the three unknowns \( x_1, x_2, x_3 \) for point \( \mathbf{x} \) that lies on each of the planes
Given: three planes

\[ x_1 + x_3 = 1 \quad x_3 = 1 \quad x_2 = 2 \]

The linear system is

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}
\]

Solving it by Gauss elimination:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
2 \\
1
\end{bmatrix}
\]
Intersecting Three Planes

Given: three planes with normal vectors

\[ \mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{n}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ \mathbf{n}_2 = \mathbf{n}_1 + \mathbf{n}_3 \]

⇒ planes are linearly dependent
⇒ do not intersect in one point
Intersecting Two Planes

Intersecting two planes is harder than intersecting three planes

Given: two planes

\[ \mathbf{n} \cdot \mathbf{x} + c = 0 \]
\[ \mathbf{m} \cdot \mathbf{x} + d = 0 \]

Find their intersection — a line

Solution of the form

\[ \mathbf{x}(t) = \mathbf{p} + t \mathbf{v} \]
Intersecting Two Planes

\( \mathbf{v} \) lies in both planes
\( \Rightarrow \) perpendicular to both normals:
\[
\mathbf{v} = \mathbf{n} \wedge \mathbf{m}
\]

Construct an auxiliary plane that intersects both given planes:
\[
\mathbf{v} \cdot \mathbf{x} = 0
\]
Passes through origin; normal \( \mathbf{v} \)
\( \Rightarrow \) perpendicular to intersection line

Next: solve the three-plane intersection problem for \( \mathbf{p} \)
Creating Orthonormal Coordinate Systems

Given: three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Find: a close orthonormal set of vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$

Solution: Gram-Schmidt method

Key: Orthogonal projections and orthogonal components

$V_i$: subspace formed by $\mathbf{v}_i$
$V_{12}$: subspace formed by $\mathbf{v}_1, \mathbf{v}_2$

Notational shorthand: normalize a vector $\mathbf{w}$ write $\mathbf{w}/\|\cdot\|$
Creating Orthonormal Coordinate Systems

$$b_1 = \frac{v_1}{\| \cdot \|}$$

Create $b_2$ from component of $v_2$ that is orthogonal to the subspace $V_1$ ⇒ normalize $(v_2 - \text{proj}_{V_1} v_2)$:

$$b_2 = \frac{v_2 - (v_2 \cdot b_1)b_1}{\| \cdot \|}$$

Create $b_3$ from component of $v_3$ that is orthogonal to the subspace $V_{12}$ ⇒ normalize $(v_3 - \text{proj}_{V_{12}} v_3)$

Separate the projection into the sum of a projection onto $V_1$ and onto $V_2$:

$$b_3 = \frac{v_3 - (v_3 \cdot b_1)b_1 - (v_3 \cdot b_2)b_2}{\| \cdot \|}$$
Creating Orthonormal Coordinate Systems

Example: Given:

\[ \mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \]
\[ \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \]
\[ \mathbf{v}_3 = \begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix} \]

\[ \mathbf{b}_1 = \frac{\mathbf{v}_1}{\| \mathbf{v}_1 \|} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \]
Creating Orthonormal Coordinate Systems

Projection of $v_2$ into subspace $V_1$:

$$u = \text{proj}_{V_1} v_2$$

$$= (\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$b_2$: component of $v_2$ orthogonal to $u$

$$b_2 = \frac{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}{\| \cdot \|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
Creating Orthonormal Coordinate Systems

Projection of $v_3$ into subspace $V_{12}$

$$w = \text{proj}_{V_{12}} v_3$$

$$= \text{proj}_{V_1} v_3 + \text{proj}_{V_2} v_3$$

$$= (\begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}) + (\begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix})$$

$$= \begin{bmatrix} 2 \\ -0.5 \\ 0 \end{bmatrix}$$

$b_3$: component of $v_3$ orthogonal to $w$

$$b_3 = \frac{\begin{bmatrix} 2.0 \\ -0.5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -0.5 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 2.0 \\ -0.5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -0.5 \\ 0 \end{bmatrix} \right\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
Creating Orthonormal Coordinate Systems

In 3D: Gram-Schmidt method requires more multiplications and additions than simply applying the cross product repeatedly.

Example: \( \mathbf{b}_3 = \mathbf{b}_1 \wedge \mathbf{b}_2 \) — and get a normalized vector for free.

Real advantage of the Gram-Schmidt method is for dimensions higher than three — where we don’t have cross products.

— Understanding the process in 3D makes the \( n \)-dimensional formulas easier to follow.
- distance between a point and plane
- distance between two lines
- plane and line intersection
- triangle and line intersection
- reflection vector
- Householder matrix
- intersection of three planes
- intersection of two planes
- Gram-Schmidt orthonormalization