Practical Linear Algebra: A Geometry Toolbox
Third edition

Chapter 11: Interactions in 3D

Gerald Farin & Dianne Hansford

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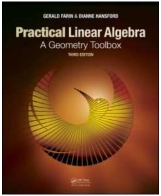
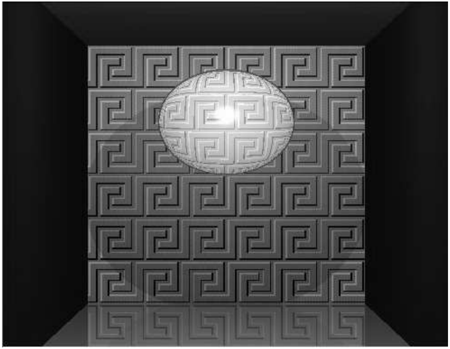


Outline

Ray tracing: 3D intersections key for rendering a raytraced image

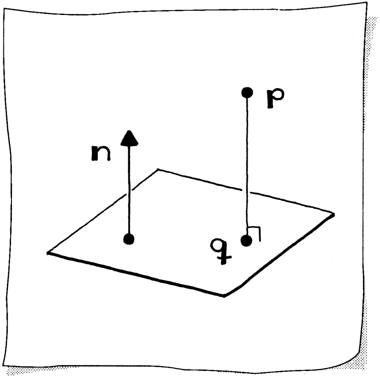


Points, lines, and planes:
basic 3D geometry building blocks

Build real objects
⇒ compute with these building blocks
— Example: intersection

(Description of the ray tracing technique is in this chapter)

Distance Between a Point and a Plane



Given:

— Plane $\mathbf{n} \cdot \mathbf{x} + c = 0$

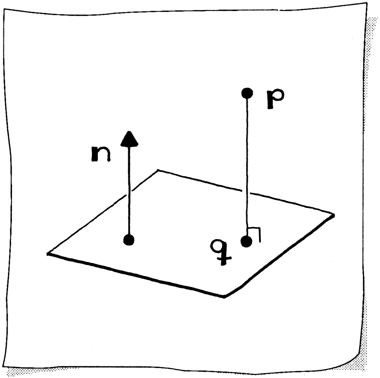
— Point $\mathbf{p}$

What is $\mathbf{p}$'s distance d to the plane?

What is $\mathbf{p}$'s closest point $\mathbf{q}$ on the plane?

Similar to the *foot of a point* from Chapter 3 2D Lines

Distance Between a Point and a Plane



Vector $\mathbf{p} - \mathbf{q}$ must be perpendicular to the plane
$\Rightarrow$ parallel to the plane's normal $\mathbf{n}$

$$\mathbf{p} = \mathbf{q} + t\mathbf{n};$$

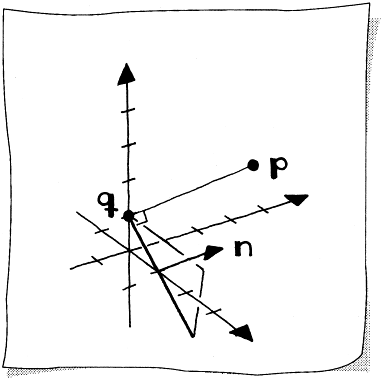
Goal: find t
$\mathbf{q}$ satisfies the plane equation:

$$\mathbf{n} \cdot [\mathbf{p} - t\mathbf{n}] + c = 0$$

$$t = \frac{c + \mathbf{n} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{n}}$$

$t = 0 \Rightarrow \mathbf{p}$ is on the plane

Example: point and a plane



Plane

$$x_1 + x_2 + x_3 - 1 = 0$$

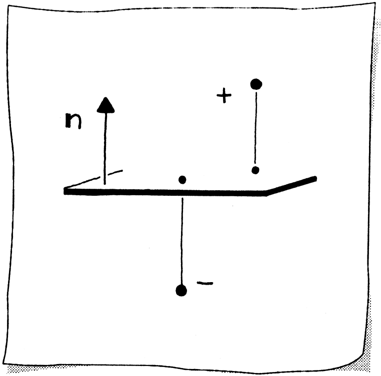
and the point

$$\mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$t = \frac{c + \mathbf{n} \cdot \mathbf{p}}{\mathbf{n} \cdot \mathbf{n}} = 2$$

$$\mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - 2 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Distance Between a Point and a Plane



Distance of **p** to the plane:
$d = \|\mathbf{p} - \mathbf{q}\| = \|t\mathbf{n}\| = t\|\mathbf{n}\|$

If **n** normalized: $\|\mathbf{p} - \mathbf{q}\| = t$ and

$$d = c + \mathbf{n} \cdot \mathbf{p}$$

If $t > 0$ then **n** points towards **p**

If $t < 0$ then **n** points away from **p**

If a point is very close to a plane can be numerically hard to decide which side it is on

Distance Between Two Lines

Two 3D lines typically do not meet — such lines are called *skew*
What is the *distance* between the lines?

$$\mathbf{l}_1 : \mathbf{x}_1(s_1) = \mathbf{p}_1 + s_1\mathbf{v}_1 \qquad \mathbf{l}_2 : \mathbf{x}_2(s_2) = \mathbf{p}_2 + s_2\mathbf{v}_2$$

$\mathbf{x}_1$: the point on $\mathbf{l}_1$ closest to $\mathbf{l}_2$
$\mathbf{x}_2$: the point on $\mathbf{l}_2$ closest to $\mathbf{l}_1$
Vector $\mathbf{x}_2 - \mathbf{x}_1$ is perpendicular to both $\mathbf{l}_1$ and $\mathbf{l}_2$:

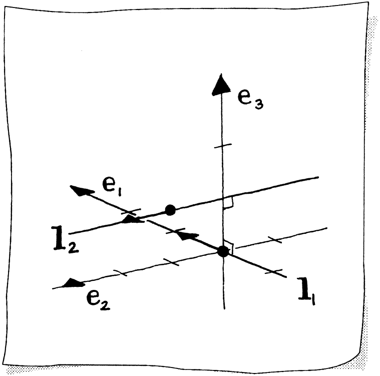
$$[\mathbf{x}_2 - \mathbf{x}_1]\mathbf{v}_1 = 0$$
$$[\mathbf{x}_2 - \mathbf{x}_1]\mathbf{v}_2 = 0$$

or

$$[\mathbf{p}_2 - \mathbf{p}_1]\mathbf{v}_1 = s_1\mathbf{v}_1 \cdot \mathbf{v}_1 - s_2\mathbf{v}_1 \cdot \mathbf{v}_2$$
$$[\mathbf{p}_2 - \mathbf{p}_1]\mathbf{v}_2 = s_1\mathbf{v}_1 \cdot \mathbf{v}_2 - s_2\mathbf{v}_2 \cdot \mathbf{v}_2$$

Two equations in the two unknowns s_1 and s_2

Example: distance between two lines



$$\mathbf{l}_1 : \mathbf{x}_1(s_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{l}_2 : \mathbf{x}_2(s_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution $s_1 = 0$ and $s_2 = -1 \Rightarrow$

$$\mathbf{x}_1(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(-1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Distance Between Two Lines

Two 3D lines intersect if $\mathbf{x}_1 = \mathbf{x}_2$
Floating point calculations $\Rightarrow$ round-off error
$\Rightarrow$ accept closeness within a tolerance: $\|\mathbf{x}_1 - \mathbf{x}_2\|^2 <$ tolerance

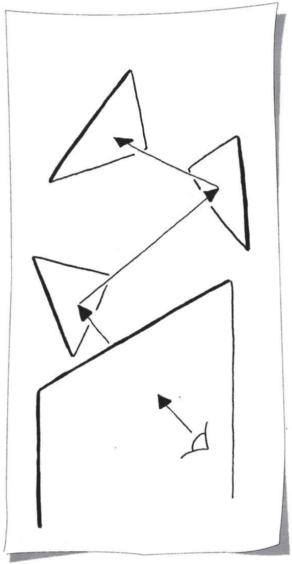
A condition for two 3D lines to intersect:
$\mathbf{v}_1, \mathbf{v}_2, \mathbf{p}_2 - \mathbf{p}_1$ must be coplanar or linearly dependent

$$\det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{p}_2 - \mathbf{p}_1] = 0$$

Numerical viewpoint: safer to compare the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$

— Distance tolerance easier to prescribed than volume tolerance

Lines and Planes: Intersections



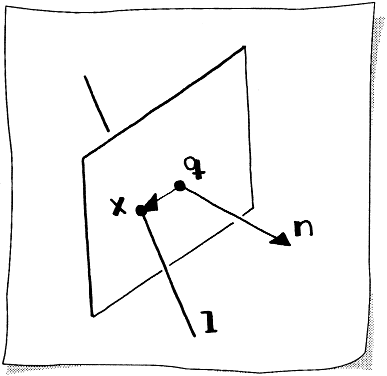
Ray tracing: basic techniques in computer graphics for creating an image

Scene given as an assembly of planes

Lighting computed by tracing light rays through the scene

Ray intersects a plane, it is reflected, then it intersects the next plane, etc.

Lines and Planes: Intersections



Given:

— Plane **P** (point **q** and normal **n**)

— Line **l** (point **p** and vector **v**)

What is their *intersection point* **x**?

On plane: $[\mathbf{x} - \mathbf{q}] \cdot \mathbf{n} = 0$

On line: $\mathbf{x} = \mathbf{p} + t\mathbf{v}$

Find t

$$[\mathbf{p} + t\mathbf{v} - \mathbf{q}] \cdot \mathbf{n} = 0$$

$$[\mathbf{p} - \mathbf{q}] \cdot \mathbf{n} + t\mathbf{v} \cdot \mathbf{n} = 0$$

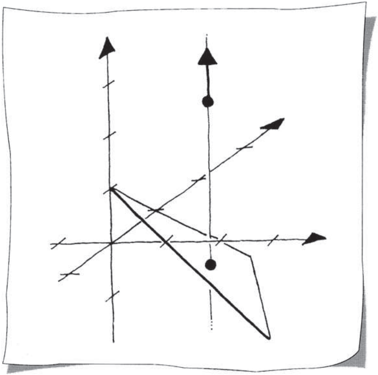
$$t = \frac{[\mathbf{q} - \mathbf{p}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}$$

$$\mathbf{x} = \mathbf{p} + \frac{[\mathbf{q} - \mathbf{p}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}$$

Caution: $\mathbf{v} \cdot \mathbf{n}$ might be small or zero

— Geometric interpretation?

Example: Intersecting a line and a plane



Plane $x_1 + x_2 + x_3 - 1 = 0$

Line

$$\mathbf{p}(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Need a point **q** on the plane:
set $x_1 = x_2 = 0$ and solve for x_3
— resulting in $x_3 = 1$

$$t = \frac{[\mathbf{q} - \mathbf{p}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} = -3$$

Intersection point:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Lines and Planes: Intersections

Intersecting a plane with a line when given plane is in *parametric form*
Unknown intersection point $\mathbf{x}$ must satisfy

$$\mathbf{x} = \mathbf{q} + u_1\mathbf{r}_1 + u_2\mathbf{r}_2$$

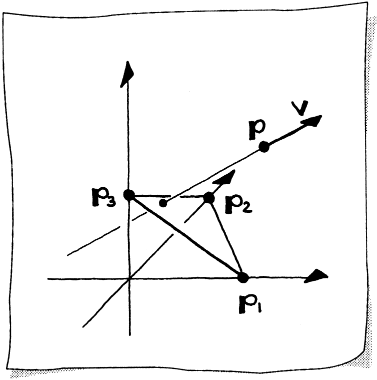
$\mathbf{x}$ is also on the line $\mathbf{l}$:

$$\mathbf{p} + t\mathbf{v} = \mathbf{q} + u_1\mathbf{r}_1 + u_2\mathbf{r}_2$$

Three equations in three unknowns

$$\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & -\mathbf{v}\end{bmatrix}\begin{bmatrix}u_1 \\ u_2 \\ t\end{bmatrix} = \begin{bmatrix}\mathbf{p} - \mathbf{q}\end{bmatrix}$$

Ray and 3D triangle intersection:
record intersection interior to triangle



Triangle: $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ Ray: $\mathbf{p}$, $\mathbf{v}$
Find intersection $\Rightarrow$ find t to satisfy

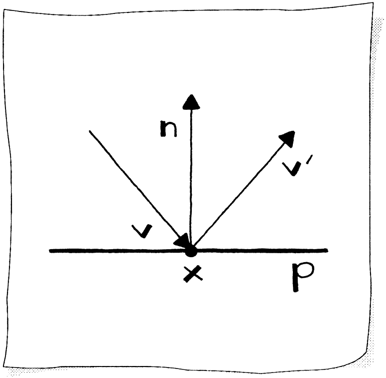
$$\mathbf{p}+t\mathbf{v} = \mathbf{p}_1+u_1(\mathbf{p}_2-\mathbf{p}_1)+u_2(\mathbf{p}_3-\mathbf{p}_1)$$

Linear system:
3 equations, 3 unknowns t, u_1, u_2
Intersection point
$= u_1\mathbf{p}_2 + u_2\mathbf{p}_3 + (1 - u_1 - u_2)\mathbf{p}_1$
— *barycentric coordinates*
Intersection inside the triangle if

$$0 \le u_1, u_2 \le 1$$

$$u_1 + u_2 \le 1$$

Reflections



Given: point $\mathbf{x}$ on a plane $\mathbf{P}$
and an "incoming" direction $\mathbf{v}$

What is the reflected or "outgoing" direction $\mathbf{v}'$?
(Assume $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{n}$ unit length)

$\mathbf{n}$ is the *angle bisector* of $\mathbf{v}$ and $\mathbf{v}'$

$$-\mathbf{v} \cdot \mathbf{n} = \mathbf{v}' \cdot \mathbf{n}$$

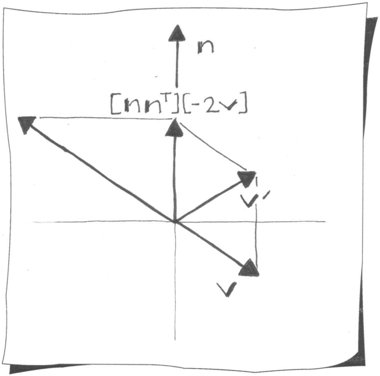
Symmetry property: $c\mathbf{n} = \mathbf{v}' - \mathbf{v}$

$$-\mathbf{v} \cdot \mathbf{n} = [c\mathbf{n} + \mathbf{v}] \cdot \mathbf{n}$$

Solve for $c = -2\mathbf{v} \cdot \mathbf{n}$

$$\mathbf{v}' = \mathbf{v} - [2\mathbf{v} \cdot \mathbf{n}]\mathbf{n}$$

Reflections



$$\mathbf{v}' = \mathbf{v} - [2\mathbf{v}\cdot\mathbf{n}]\mathbf{n}$$
$$= \mathbf{v} - 2[\mathbf{v}^{\mathrm{T}}\mathbf{n}]\mathbf{n}$$
$$= \mathbf{v} - 2[\mathbf{n}\mathbf{n}^{\mathrm{T}}]\mathbf{v}$$

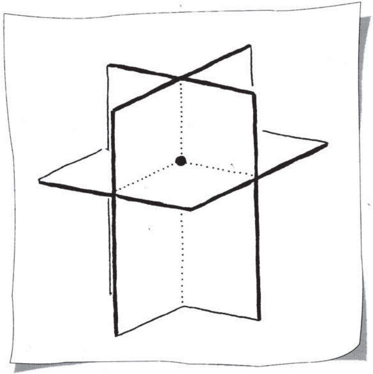
$\mathbf{n}\mathbf{n}^{\mathrm{T}}$ is a projection matrix
— orthogonal
— symmetric
— *dyadic* (rank one)
Reflection as a linear map: $\mathbf{v}' = H\mathbf{v}$

$$H = I - 2\mathbf{n}\mathbf{n}^{\mathrm{T}}$$

Householder matrix H

Chapter 13 — The Householder Method

Intersecting Three Planes



Given: three planes

$$\mathbf{n}_1 \cdot \mathbf{x} + c_1 = 0$$
$$\mathbf{n}_2 \cdot \mathbf{x} + c_2 = 0$$
$$\mathbf{n}_3 \cdot \mathbf{x} + c_3 = 0$$

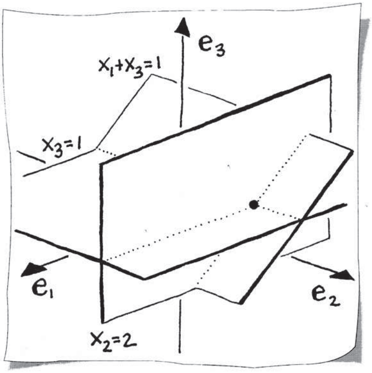
Where do they intersect?

Plane equations into matrix form:

$$\begin{bmatrix} \mathbf{n}_1^{\mathrm{T}} \\ \mathbf{n}_2^{\mathrm{T}} \\ \mathbf{n}_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix}$$

Solve three equations in the three unknowns x_1, x_2, x_3 for point $\mathbf{x}$ that lies on each of the planes

Intersecting Three Planes



Given: three planes

$$x_1 + x_3 = 1 \quad x_3 = 1 \quad x_2 = 2$$

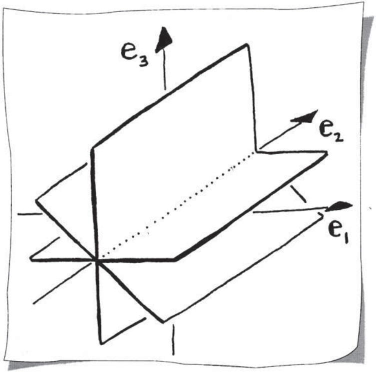
The linear system is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Solving it by Gauss elimination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Intersecting Three Planes



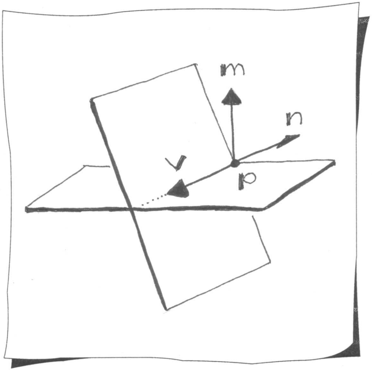
Given: three planes with normal vectors

$$\mathbf{n}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{n}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{n}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{n}_2 = \mathbf{n}_1 + \mathbf{n}_3$
$\Rightarrow$ planes are linearly dependent
$\Rightarrow$ do not intersect in one point

Intersecting Two Planes

Intersecting two planes is harder than intersecting three planes



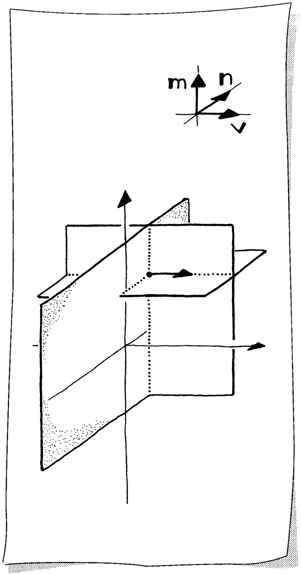
Given: two planes

$$\mathbf{n} \cdot \mathbf{x} + c = 0$$
$$\mathbf{m} \cdot \mathbf{x} + d = 0$$

Find their intersection — a line
Solution of the form

$$\mathbf{x}(t) = \mathbf{p} + t\mathbf{v}$$

Intersecting Two Planes



v lies in both planes
⇒ perpendicular to both normals:

$$\mathbf{v} = \mathbf{n} \wedge \mathbf{m}$$

Construct an auxiliary plane that intersects both given planes:

$$\mathbf{v} \cdot \mathbf{x} = 0$$

Passes through origin; normal **v**
⇒ perpendicular to intersection line

Next: solve the three-plane intersection problem for **p**

Creating Orthonormal Coordinate Systems

Given: three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Find: a close *orthonormal* set of vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$

Solution: Gram-Schmidt method

Key: Orthogonal projections and orthogonal components

V_i: subspace formed by $\mathbf{v}_i$
V_{12}: subspace formed by $\mathbf{v}_1, \mathbf{v}_2$

Notational shorthand: normalize a vector $\mathbf{w}$ write $\mathbf{w}/\|\cdot\|$

Creating Orthonormal Coordinate Systems

$$\mathbf{b}_1 = \frac{\mathbf{v}_1}{\|\cdot\|}$$

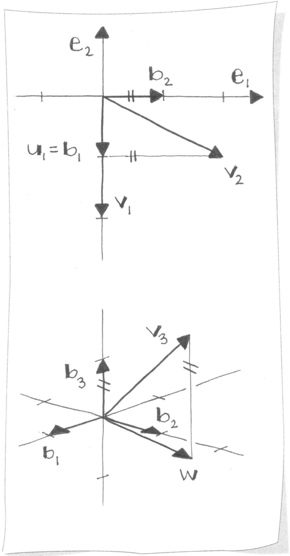
Create $\mathbf{b}_2$ from component of $\mathbf{v}_2$ that is orthogonal to the subspace V_1
$\Rightarrow$ normalize $(\mathbf{v}_2 - \mathrm{proj}_{V_1}\mathbf{v}_2)$:

$$\mathbf{b}_2 = \frac{\mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{b}_1)\mathbf{b}_1}{\|\cdot\|}$$

Create $\mathbf{b}_3$ from component of $\mathbf{v}_3$ that is orthogonal to the subspace V_{12}
$\Rightarrow$ normalize $(\mathbf{v}_3 - \mathrm{proj}_{V_{12}}\mathbf{v}_3)$
Separate the projection into the sum of a projection onto V_1 and onto V_2:

$$\mathbf{b}_3 = \frac{\mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{b}_1)\mathbf{b}_1 - (\mathbf{v}_3 \cdot \mathbf{b}_2)\mathbf{b}_2}{\|\cdot\|}$$

Creating Orthonormal Coordinate Systems



Example: Given:

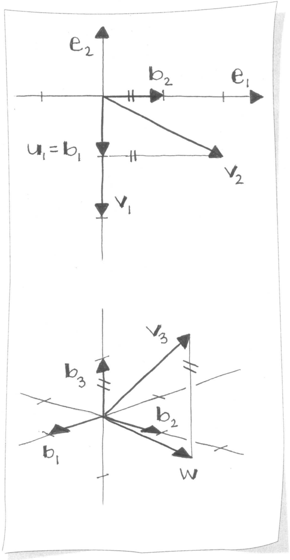
$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix}$$

$$\mathbf{b}_1 = \frac{\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}}{\| \cdot \|} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Creating Orthonormal Coordinate Systems



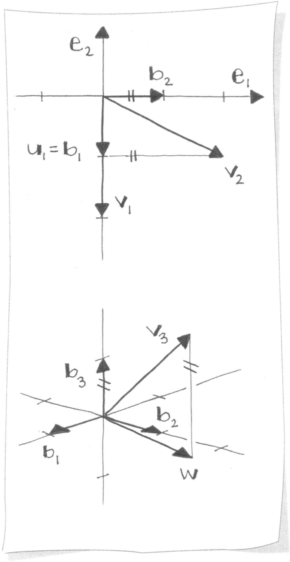
Projection of $\mathbf{v}_2$ into subspace V_1:

$$\mathbf{u} = \text{proj}_{V_1} \mathbf{v}_2$$
$$= \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$\mathbf{b}_2$: component of $\mathbf{v}_2$ orthogonal to $\mathbf{u}$

$$\mathbf{b}_2 = \frac{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}{\|\cdot\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Creating Orthonormal Coordinate Systems



Projection of $\mathbf{v}_3$ into subspace V_{12}

$$
\begin{aligned}
\mathbf{w} &= \operatorname{proj}_{V_{12}} \mathbf{v}_3 \\
&= \operatorname{proj}_{V_1} \mathbf{v}_3 + \operatorname{proj}_{V_2} \mathbf{v}_3 \\
&= \left(\begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 2 \\ -0.5 \\ 2 \end{bmatrix} \right. \\
&= \begin{bmatrix} 2 \\ -0.5 \\ 0 \end{bmatrix}
\end{aligned}
$$

$\mathbf{b}_3$: component of $\mathbf{v}_3$ orthogonal to $\mathbf{w}$

$$
\mathbf{b}_3 = \frac{\begin{bmatrix} 2.0 \\ -0.5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -0.5 \\ 0 \end{bmatrix}}{\| \cdot \|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Creating Orthonormal Coordinate Systems

In 3D: Gram-Schmidt method requires more multiplications and additions than simply applying the cross product repeatedly

Example: $\mathbf{b}_3 = \mathbf{b}_1 \wedge \mathbf{b}_2$ — and get a normalized vector for free

Real advantage of the Gram-Schmidt method is for dimensions higher than three
— where we don't have cross products
— Understanding the process in 3D makes the *n*-dimensional formulas easier to follow

WYSK

- distance between a point and plane
- distance between two lines
- plane and line intersection
- triangle and line intersection
- reflection vector
- Householder matrix
- intersection of three planes
- intersection of two planes
- Gram-Schmidt orthonormalization