Outline

1. Introduction to General Linear Spaces
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3. Linear Maps
4. Inner Products
5. Gram-Schmidt Orthonormalization
6. A Gallery of Spaces
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General Linear Spaces

All cubic polynomials over the interval \([0,1]\) form a linear space
Some elements illustrated

Linear space = vector space
Chapters 4 and 9: examined
properties in 2D and 3D

Here: higher dimensions
— Spaces can be abstract
— Powerful concept in dealing with
real-life problems
  • car crash simulations
  • weather forecasts
  • computer games

“General” refers to the dimension
and abstraction
Basic Properties of Linear Spaces

\( L_n: \text{linear space of dimension } n \)

Elements of \( L_n \) are vectors
— Denoted by boldface letters such as \( \mathbf{u} \)

Two operations defined on the elements of \( L_n \):
— Addition
— Multiplication by a scalar

**Linearity property**
Any *linear combination* of vectors results in a vector in the same space

\[
\mathbf{w} = s \mathbf{u} + t \mathbf{v}
\]

Both \( s \) and \( t \) may be zero \( \Rightarrow \) every linear space has a zero vector in it
Basic Properties of Linear Spaces

Generalize linear spaces: include new kinds of vectors
— Objects in the linear space are not always in traditional vector format
— Key: the linearity property

Example: \( \mathbb{R}^2 \)

Elements of space: \( u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( v = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \)

\[ \Rightarrow w = 2u + v = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \] is also in \( \mathbb{R}^2 \)

Example: Linear space \( \mathcal{M}_{2 \times 2} \) – the set of all \( 2 \times 2 \) matrices
— Rules of matrix arithmetic guarantee the linearity property

Example: \( \mathcal{V}_2 \) – all vectors \( w \) in \( \mathbb{R}^2 \) that satisfy \( w_2 \geq 0 \)
— \( e_1 \) and \( e_2 \) live in \( \mathcal{V}_2 \) — Is this a linear space?

No: \( v = 0 \times e_1 + -1 \times e_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) which is not in \( \mathcal{V}_2 \)
Basic Properties of Linear Spaces

In $\mathcal{L}_n$ define a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ where $1 \leq r \leq n$

Vectors are \textbf{linearly independent} means

$$\mathbf{v}_1 = s_2 \mathbf{v}_2 + s_3 \mathbf{v}_3 + \ldots + s_r \mathbf{v}_r$$

Will \textit{not} have a solution set $s_2, \ldots, s_r$

$\Rightarrow$ Zero vector can only be expressed in a trivial manner:

If $\mathbf{0} = s_1 \mathbf{v}_1 + \ldots + s_r \mathbf{v}_r$ then $s_1 = \ldots = s_r = 0$

If the zero vector \textit{can} be expressed as a nontrivial combination of $r$ vectors then these vectors are \textbf{linearly dependent}
Basic Properties of Linear Spaces

Subspace of $\mathcal{L}_n$ of dimension $r$:
Formed from all linear combinations of linearly independent $v_1, \ldots, v_r$
$\Rightarrow$ Subspace is spanned by $v_1, \ldots, v_r$

If this subspace equals whole space $\mathcal{L}_n$ then $v_1, \ldots, v_n$ a basis for $\mathcal{L}_n$

If $\mathcal{L}_n$ is a linear space of dimension $n$
then any $n + 1$ vectors in it are linearly dependent

Example: $\mathbb{R}^3$ and basis vectors $e_1, e_2, e_3$

$$v = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is also in } \mathbb{R}^3$$

The four vectors $v, e_1, e_2, e_3$ are linearly dependent

Any one of four vectors forms a one-dimensional subspace of $\mathbb{R}^3$
Any two vectors here form a two-dimensional subspace of $\mathbb{R}^3$
Example: \( \mathbb{R}^4 \)

\[
\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}
\]

These vectors are linearly dependent since

\[
\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_3 \quad \text{or} \quad 0 = \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3
\]

Set \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) contains only two linearly independent vectors

\( \Rightarrow \) Any two of them spans a subspace of \( \mathbb{R}^4 \) of dimension two
Basic Properties of Linear Spaces

Example: \( \mathbb{R}^3 \)

\[
\begin{align*}
v_1 &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} & v_2 &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} & v_3 &= \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} & v_4 &= \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}
\end{align*}
\]

These four vectors are linearly dependent since

\[ v_3 = -v_1 + 2v_2 + v_4 \]

Any set of three of these vectors is a basis for \( \mathbb{R}^3 \)
Linear Maps

\( A : \mathcal{L}_n \rightarrow \mathcal{L}_m \) — The linear map \( A \) that transforms \( \mathcal{L}_n \) to \( \mathcal{L}_m \)

\( A \) preserves linear relationships

Preimage \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) in \( \mathcal{L}_n \) mapped to image \( A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3 \) in \( \mathcal{L}_m \)

\[ \mathbf{v}_1 = \alpha \mathbf{v}_2 + \beta \mathbf{v}_3 \quad \Rightarrow \quad A\mathbf{v}_1 = \alpha A\mathbf{v}_2 + \beta A\mathbf{v}_3 \]

Maps without this property: nonlinear maps

Linear map: \( m \times n \) matrix \( A \)

\( \mathbf{v} \) in \( \mathcal{L}_n \) → \( \mathbf{v}' \) in \( \mathcal{L}_m \)  \( \Rightarrow \) \( \mathbf{v}' = A\mathbf{v} \)

\( A : [\mathbf{e}_1, \ldots, \mathbf{e}_n]\)-system → \( [\mathbf{a}_1, \ldots, \mathbf{a}_n]\)-system

\( \Rightarrow \)

\[ \mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n \quad \text{is in the column space of} \ A \]
Linear Maps

Example: \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
2 & 2
\end{bmatrix}
\]

Given vectors in \( \mathbb{R}^2 \)

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]

mapped to vectors in \( \mathbb{R}^3 \)

\[
\hat{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \hat{\mathbf{v}}_3 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}
\]

\( \mathbf{v}_i \) are *linearly dependent* since \( \mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2 \)

Linear maps preserve linear relationships \( \Rightarrow \mathbf{v}_3' = 2\mathbf{v}_1' + \mathbf{v}_2' \)
Matrix rank

$m \times n$ matrix can be at most of rank $k = \min\{m, n\}$

Rank equals number of linearly independent column vectors

If $\text{rank}(A) = \min\{m, n\}$ ⇒ full rank

If $\text{rank}(A) < \min\{m, n\}$ ⇒ rank deficient

Linear map can never *increase* dimension

— Possible to map $\mathcal{L}_n$ to higher-dimensional space $\mathcal{L}_m$
  Images of $\mathcal{L}_n$’s $n$ basis vectors will span
  a subspace of $\mathcal{L}_m$ of dimension at most $n$

(See last Example)

How to identify rank?

Perform forward elimination until matrix in upper triangular form

— $k$ nonzero rows ⇒ rank is $k$
Rank scenarios for an \( m \times n \) matrix
Matrices in upper triangular form

\[
m < n \quad m = n \quad m > n
\]

Top row: full rank matrices
Bottom row: rank deficient matrices
Example: Determine the rank of the matrix

\[
\begin{bmatrix}
1 & 3 & 4 \\
0 & 1 & 2 \\
1 & 2 & 2 \\
-1 & 1 & 1
\end{bmatrix}
\]

Forward elimination \( \Rightarrow \)

\[
\begin{bmatrix}
1 & 3 & 4 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

One row of zeroes: matrix has rank 3 — full rank since \( \min\{4, 3\} = 3 \)

Example: Determine the rank of the matrix

\[
\begin{bmatrix}
1 & 3 & 4 \\
0 & 1 & 2 \\
1 & 2 & 2 \\
0 & 1 & 2
\end{bmatrix}
\]

Forward elimination \( \Rightarrow \)

\[
\begin{bmatrix}
1 & 3 & 4 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Matrix has rank 2 — rank deficient
Review features of linear maps from earlier chapters

\( n \times n \) matrix \( A \) that is rank \( n \) is invertible
\[ \Rightarrow \text{inverse matrix } A^{-1} \text{ exists} \]

If \( A \) is invertible then it does not reduce dimension
\[ \Rightarrow \text{Determinant is nonzero} \]

- Measures volume of \( nD \) parallelepiped defined by columns vectors
- Computed by transforming matrix to upper triangular
  (via shears/forward elimination)
  Then the determinant is the product of the diagonal elements
  (pivoting: careful of sign)
Inner Products

**Inner product**: a map from $\mathcal{L}_n$ to the reals $\mathbb{R}$ — denoted as $\langle \mathbf{v}, \mathbf{w} \rangle$

Properties:

- **Symmetry**: $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$
- **Homogeneity**: $\langle \alpha \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{w}, \mathbf{v} \rangle$
- **Additivity**: $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}$ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$
- **Positivity**: $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

Homogeneity and additivity properties combined:

$$\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$$

**Example**: the *dot product* $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n$

**Inner product space**: a linear space with an inner product
**Example:** Define a “test” inner product in $\mathbb{R}^2$

$$\langle \mathbf{v}, \mathbf{w} \rangle = 4v_1w_1 + 2v_2w_2$$

Compare it to the dot product:

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 4(1)(0) + 2(0)(1) = 0 \quad \text{e}_1 \cdot \text{e}_2 = 0$$

Let $\mathbf{r} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (unit vector)

$$\langle \mathbf{e}_1, \mathbf{r} \rangle = 4(1)(\frac{1}{\sqrt{2}}) + 2(0)(\frac{1}{\sqrt{2}}) = \frac{4}{\sqrt{2}} \quad \text{e}_1 \cdot \mathbf{r} = \frac{1}{\sqrt{2}}$$
Does the test inner product satisfy the necessary properties?

Symmetry: \( \langle \mathbf{v}, \mathbf{w} \rangle = 4v_1w_1 + 2v_2w_2 = 4w_1v_1 + 2w_2v_2 = \langle \mathbf{w}, \mathbf{v} \rangle \)

Homogeneity: \( \langle \alpha \mathbf{v}, \mathbf{w} \rangle = 4(\alpha v_1)w_1 + 2(\alpha v_2)w_2 = \alpha(4v_1w_1 + 2v_2w_2) = \alpha \langle \mathbf{v}, \mathbf{w} \rangle \)

Additivity: \( \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 4(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \)

\[ = (4u_1w_1 + 2u_2w_2) + (4v_1w_1 + 2v_2w_2) \]

\[ = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \]

Positivity: \( \langle \mathbf{v}, \mathbf{v} \rangle = 4v_1^2 + 2v_2^2 \geq 0 \) and \( \langle \mathbf{v}, \mathbf{v} \rangle = 0 \) iff \( \mathbf{v} = \mathbf{0} \)

Usefulness of this inner product? But it does satisfy the properties!
Inner Products

Length

2-norm or Euclidean norm: \( \|v\|_2 = \sqrt{\langle v, v \rangle} \)
(“Usual” norm ⇒ subscript typically omitted)

Distance between two vectors

\[
\text{dist}(u, v) = \sqrt{\langle u - v, u - v \rangle} = \|u - v\|
\]

Example: the dot product in \( \mathbb{R}^n \)

\[
\|v\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}
\]

\[
\text{dist}(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}
\]
Norm and distance for two inner products

**Test inner product**

\[ \langle v, w \rangle = 4v_1w_1 + 2v_2w_2 \]

\[ \| e_1 \| = \sqrt{\langle e_1, e_1 \rangle} = 4(1)^2 + 2(0)^2 = 4 \]

\[ \text{dist}(e_1, e_2) = \sqrt{4(1 - 0)^2 + 2(0 - 1)^2} = \sqrt{6} \]

**Dot product**

\[ \langle v, w \rangle = v_1w_1 + v_2w_2 \]

\[ \| e_1 \| = 1 \]

\[ \text{dist}(e_1, e_2) = \sqrt{2} \]
Inner Products

Black: dot product  Gray: test inner product  \[ \langle \mathbf{v}, \mathbf{w} \rangle = 4v_1 w_1 + 2v_2 w_2 \]

Unit vector \( \mathbf{r} \) rotated \([0, 2\pi]\)

Left: inner product \( \mathbf{e}_1 \cdot \mathbf{r} \) and \( \langle \mathbf{e}_1, \mathbf{r} \rangle \)

Middle: length \( \sqrt{\mathbf{r} \cdot \mathbf{r}} \) and \( \sqrt{\langle \mathbf{r}, \mathbf{r} \rangle} \)

Right: distance \( \sqrt{(\mathbf{e}_1 - \mathbf{r}) \cdot (\mathbf{e}_1 - \mathbf{r})} \) and \( \sqrt{\langle (\mathbf{e}_1 - \mathbf{r}), (\mathbf{e}_1 - \mathbf{r}) \rangle} \)
Orthogonality: $\langle v, w \rangle = 0$ for $v, w$ in $\mathcal{L}_n$

Orthogonal basis: $v_1, \ldots, v_n$ form a basis for $\mathcal{L}_n$ and all $v_i$ are mutually orthogonal: $\langle v_i, v_j \rangle = 0$ for $i \neq j$

And if all $v_i$ are unit length: $\|v_i\| = 1$

they form an orthonormal basis

The *Gram-Schmidt method*:
— Basis of a linear space $\Rightarrow$ an orthonormal basis
— See the next Section
Inner Products

Cauchy-Schwartz inequality — in the context of inner product spaces

$$\langle v, w \rangle^2 \leq \langle v, v \rangle \langle w, w \rangle$$

Equality holds if and only if $v$ and $w$ linearly dependent

Restate the Cauchy-Schwartz inequality

$$\langle v, w \rangle^2 \leq \|v\|^2 \|w\|^2$$

$$\left(\frac{\langle v, w \rangle}{\|v\| \|w\|}\right)^2 \leq 1$$

$$-1 \leq \frac{\langle v, w \rangle}{\|v\| \|w\|} \leq 1$$

Angle $\theta$ between $v$ and $w$

$$\cos \theta = \frac{\langle v, w \rangle}{\|v\| \|w\|}$$
Inner product properties suggest

\[ \|v\| \geq 0 \]
\[ \|v\| = 0 \text{ if and only if } v = 0 \]
\[ \|\alpha v\| = |\alpha|\|v\| \]

A fourth property is the triangle inequality:

\[ \|v + w\| \leq \|v\| + \|w\| \]

(derived from the Cauchy-Schwartz inequality in Chapter 2)
Inner Products

General definition of a projection

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ span a subspace $\mathcal{L}_k$ of $\mathcal{L}$.
If $\mathbf{v}$ is a vector not in $\mathcal{L}_k$ then

$$P\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \ldots + \langle \mathbf{v}, \mathbf{u}_k \rangle \mathbf{u}_k$$

is $\mathbf{v}$'s orthogonal projection into $\mathcal{L}_k$. 
Gram-Schmidt Orthonormalization

Every inner product space has an orthonormal basis

**Given:** orthonormal vectors $b_1, \ldots, b_r$

— Form basis of subspace $S_r$ of $\mathcal{L}_n$ where $n > r$

**Find:** $b_{r+1}$ orthogonal to the given $b_i$

Let $u$ be an arbitrary vector in $\mathcal{L}_n$, but not in $S_r$

*u's orthogonal projection* into $S_r$:

$$\hat{u} = \text{proj}_{S_r} u = \langle u, b_1 \rangle b_1 + \ldots + \langle u, b_r \rangle b_r$$

Check orthogonality: for example $\langle u - \hat{u}, b_1 \rangle = 0$

$$\langle u - \hat{u}, b_1 \rangle = \langle u, b_1 \rangle - \langle u, b_1 \rangle\langle b_1, b_1 \rangle - \ldots - \langle u, b_r \rangle\langle b_1, b_r \rangle$$

$$\Rightarrow$$

$$b_{r+1} = \frac{u - \text{proj}_{S_r} u}{\| \cdot \|}$$

Repeat to form an orthonormal basis for all of $\mathcal{L}_n$

**Key elements:** projections and vector decomposition
Gram-Schmidt Orthonormalization

$S_2$ is depicted as $\mathbb{R}^2$

Build the orthonormal basis:
Given basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathcal{L}_n$

$$
\begin{align*}
\mathbf{b}_1 &= \frac{\mathbf{v}_1}{\| \cdot \|} \\
\mathbf{b}_2 &= \frac{\mathbf{v}_2 - \text{proj}_{S_1} \mathbf{v}_2}{\| \cdot \|} = \frac{\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{b}_1 \rangle \mathbf{b}_1}{\| \cdot \|} \\
\mathbf{b}_3 &= \frac{\mathbf{v}_3 - \text{proj}_{S_2} \mathbf{v}_3}{\| \cdot \|} = \frac{\mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{b}_1 \rangle \mathbf{b}_1 - \langle \mathbf{v}_3, \mathbf{b}_2 \rangle \mathbf{b}_2}{\| \cdot \|} \\
& \vdots
\end{align*}
$$
Gram-Schmidt Orthonormalization

Example: \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \)

Form an orthonormal basis \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4 \)

\[
\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}
\]

\[
\mathbf{b}_4 = \frac{\mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{b}_1 \rangle \mathbf{b}_1 - \langle \mathbf{v}_4, \mathbf{b}_2 \rangle \mathbf{b}_2 - \langle \mathbf{v}_4, \mathbf{b}_3 \rangle \mathbf{b}_3}{\| \cdot \|} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}
\]

Check: \( \begin{vmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{vmatrix} = 1 \)
A Gallery of Spaces

Let’s highlight some special linear spaces—but there are many more!

A linear space $\mathcal{P}_n$ whose elements are all polynomials of a fixed degree $n$

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where $t$ is the independent variable of $p(t)$

— Addition in this space is coefficient by coefficient
— Multiplication in this space: polynomial times a real number

Check linearity property: $p(t) = 3 - 2t + 3t^2$ and $q(t) = -1 + t + 2t^2$
then $2p(t) + 3q(t) = 3 - t + 12t^2$ is yet another polynomial of the same degree

$\Rightarrow$ Linear map: derivative $p'$ of a degree $n$ polynomial $p$

$$p'(t) = a_1 + 2a_2 t + \ldots + na_n t^{n-1}$$

Rank of this map is $n - 1$
A Gallery of Spaces

**Example:** Two cubic polynomials

\[ p(t) = 3 - t + 2t^2 + 3t^3 \quad \text{and} \quad q(t) = 1 + t - t^3 \]

in the linear space of cubic polynomials \( P_3 \)

Let \( r(t) = 2p(t) - q(t) = 5 - 3t + 4t^2 + 7t^3 \)

\[ r'(t) = -3 + 8t + 21t^2 \]
\[ p'(t) = -1 + 4t + 9t^2 \]
\[ q'(t) = 1 - 3t^2 \]

\[ r'(t) = 2p'(t) - q'(t) \Rightarrow \text{linearity of the derivative map} \]
A Gallery of Spaces

A linear space given by the set of all real-valued continuous functions over the interval $[0, 1]$
— This space is typically named $C[0, 1]$
— The linearity condition is met:
  If $f$ and $g$ are elements of $C[0, 1]$ then $\alpha f + \beta g$ is also in $C[0, 1]$
— This is an infinite-dimensional linear space
  No finite set of functions forms a basis for $C[0, 1]$

The set of all $3 \times 3$ matrices form a linear space
— This space consists of matrices
— Linear combinations formed using standard matrix addition and multiplication with a scalar
A Gallery of Spaces

A more abstract example:
The linear space formed from the set of all linear maps from a linear space $\mathcal{L}_n$ into the reals
— Called the **dual space** $\mathcal{L}_n^*$ of $\mathcal{L}_n$
— Its dimension equals that of $\mathcal{L}_n$
— The linear maps in $\mathcal{L}_n^*$ are known as **linear functionals**

Let a fixed vector $v$ and an variable vector $u$ be in $\mathcal{L}_n$
The linear functionals defined by $\Phi_v(u) = \langle u, v \rangle$ are in $\mathcal{L}_n^*$
For any basis $b_1, \ldots, b_n$ of $\mathcal{L}_n$ define linear functionals

$$\Phi_{b_i}(u) = \langle u, b_i \rangle \quad \text{for } i = 1, \ldots, n$$

These functionals form a basis for $\mathcal{L}_n^*$
A Gallery of Spaces

Example: In $\mathbb{R}^2$ consider the fixed vector

$$v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Then $\Phi_v(u) = \langle u, v \rangle = u_1 - 2u_2$ for all vectors $u$ where $\langle \cdot, \cdot \rangle$ is the dot product.

Example: Pick $e_1, e_2$ for a basis in $\mathbb{R}^2$
The associated linear functionals are

$$\Phi_{e_1}(u) = u_1, \quad \Phi_{e_2}(u) = u_2$$

Any linear functional $\Phi$ can now be defined as

$$\Phi(u) = r_1 \Phi_{e_1}(u) + r_2 \Phi_{e_2}(u)$$

where $r_1$ and $r_2$ are scalars.
• linear space
• vector space
• dimension
• linear combination
• linearity property
• linearly independent
• subspace
• span
• linear map
• image
• preimage
• domain

• range
• rank
• full rank
• rank deficient
• inverse
• determinant
• inner product
• inner product space
• distance in an inner product space
• length in an inner product space

• orthogonal
• Gram-Schmidt method
• projection
• basis
• orthonormal
• orthogonal decomposition
• best approximation
• dual space
• linear functional