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Introduction to Breaking It Up: Triangles

2D finite element method: refinement of a triangulation based on stress and strain calculations

Triangles are as old as geometry
Of interest to the ancient Greeks

An indispensable tool in many applications
— computer graphics
— finite element analysis

Reducing the geometry to linear or piecewise planar makes computations more tractable
A triangle $T$ is given by three points

- Its vertices $p_1, p_2, p_3$
- Vertices may live in 2D or 3D

Three points define a plane

$\Rightarrow$ a triangle is a 2D element

Conventions:

- Label the $p_i$ counterclockwise
- Edge opposite point $p_i$ labeled $s_i$
Barycentric Coordinates

Invented by F. Moebius in 1827

Create a local coordinate system

Let \( p \) be an arbitrary point inside \( T \)

\[
p = up_1 + vp_2 + wp_3
\]

Right-hand side:
a combination of points
⇒ coefficients must sum to one:

\[
 u + v + w = 1
\]

As a linear system:

\[
\begin{bmatrix}
p_1 & p_2 & p_3
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= 
\begin{bmatrix}
p_1 \\
p_2 \\
p_3
\end{bmatrix}
\]
Barycentric Coordinates

Solve the $3 \times 3$ linear system with Cramer's rule

\[
\begin{align*}
u &= \frac{\text{area}(p, p_2, p_3)}{\text{area}(p_1, p_2, p_3)} \\
v &= \frac{\text{area}(p, p_3, p_1)}{\text{area}(p_1, p_2, p_3)} \\
w &= \frac{\text{area}(p, p_1, p_2)}{\text{area}(p_1, p_2, p_3)}
\end{align*}
\]

\[u = (u, v, w) \text{ called barycentric coordinates}\]

Examine the result:

- Ratios of areas
- $(u, v, w)$ sum to one $\Rightarrow$ not independent $w = 1 - u - v$
- Let $p = p_2$ $\Rightarrow$ $v = 1$ and $u = w = 0$
- If $p$ is on $s_1$ then $u = 0$
Examples of barycentric coordinates

Triangle vertices:

\[ p_1 = (1, 0, 0) \]
\[ p_2 = (0, 1, 0) \]
\[ p_3 = (0, 0, 1) \]

Even points outside of \( T \) have barycentric coordinates!
— Determinants return signed areas

Points inside \( T \): positive \((u, v, w)\)
Points outside \( T \): mixed signs
Application: **Triangle inclusion test**

**Problem:** Given a triangle $T$ and a point $p$. Is $p$ is inside $T$?

**Solution:** Compute $p$’s barycentric coordinates and check their signs!
— All the same sign then $p$ is inside $T$
— Else $p$ is outside $T$

Theoretically: one or two $(u, v, w)$ could be zero $\Rightarrow$ $p$ is on an edge

Numerically: not likely to encounter exactly zero
$\Rightarrow$ Do not test for equality
   Instead: use a zero tolerance $\epsilon$
   Is $|\text{barycentric coordinate}| < \epsilon$ ?
Barycentric Coordinates

Whole plane covered by a grid of coordinate lines

Plane divided into seven regions by the (extended) edges of $T$
Barycentric Coordinates

Example: Triangle vertices

\[
p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad p_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Points \( q, r, s \) with barycentric coordinates

\[
q \approx \left( 0, \frac{1}{2}, \frac{1}{2} \right) \quad r \approx (-1, 1, 1) \quad s \approx \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)
\]

have coordinates in the plane

\[
q = 0 \times p_1 + \frac{1}{2} \times p_2 + \frac{1}{2} \times p_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}
\]

\[
r = -1 \times p_1 + 1 \times p_2 + 1 \times p_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
s = \frac{1}{3} \times p_1 + \frac{1}{3} \times p_2 + \frac{1}{3} \times p_3 = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}
\]
Affine Invariance

Barycentric coordinates are affinely invariant

- $\hat{T}$ is an affine image of $T$
- $p \cong u$ relative to $T$
- $\hat{p}$ is an affine image of $p$

What are the barycentric coordinates of $\hat{p}$ with respect to $\hat{T}$?

*Ratios of areas* are invariant under affine maps
- Individual areas change but not the ratios

$\Rightarrow \hat{p} \cong u$ relative to $\hat{T}$
Affine Invariance

Example: Given triangle vertices

\[ \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Apply a 90° rotation

\[ R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

Resulting in \( \hat{\mathbf{p}}_i = R\mathbf{p}_i \)

Barycentric coordinates \((1/3, 1/3, 1/3)\) relative to \(T\)

\[ \mathbf{s} = \frac{1}{3} \mathbf{p}_1 + \frac{1}{3} \mathbf{p}_2 + \frac{1}{3} \mathbf{p}_3 = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathbf{s}} = R\mathbf{s} = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix} \]

Due to the affine invariance of barycentric coordinates could have found the coordinates as

\[ \hat{\mathbf{s}} = \frac{1}{3} \hat{\mathbf{p}}_1 + \frac{1}{3} \hat{\mathbf{p}}_2 + \frac{1}{3} \hat{\mathbf{p}}_3 = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix} \]
Some Special Points

The centroid $c$

Intersection of the three medians

$$c = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

Verify by writing

$$\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) = \frac{1}{3} (0, 1, 0) + \frac{2}{3} \left( \frac{1}{2}, 0, \frac{1}{2} \right)$$

⇒ centroid lies on median associated with $p_2$

— Same idea for remaining medians

Triangle is affinely related to its centroid
Some Special Points

The incenter $i = (i_1, i_2, i_3)$

Intersection of the angle bisectors $i$ is the center of the *incircle*

$s_i$: length of triangle edge opposite $p_i$

$r$: radius of the incircle

\[
   i_1 = \frac{\text{area}(i, p_2, p_3)}{\text{area}(p_1, p_2, p_3)}
\]

Use “1/2 base times height” rule

\[
   i_1 = \frac{rs_1}{rs_1 + rs_2 + rs_3}
\]

\[
   i_1 = \frac{s_1}{c} \quad i_2 = \frac{s_2}{c} \quad i_3 = \frac{s_3}{c}
\]

$c = s_1 + s_2 + s_3$ is circumference of $T$

Triangle is *not* affinely related to its incenter
Some Special Points

The **circumcenter** \( cc \)

Circle through \( T \)'s vertices called the **circumcircle**

Center of the circumcircle is the circumcenter
- Intersection of the edge bisectors
- Might not be inside the triangle
Some Special Points

The barycentric coordinates \((cc_1, cc_2, cc_3)\) of the circumcenter

\[
cc_1 = \frac{d_1(d_2 + d_3)}{D} \quad cc_2 = \frac{d_2(d_1 + d_3)}{D} \quad cc_3 = \frac{d_3(d_1 + d_2)}{D}
\]

\[
d_1 = (p_2 - p_1) \cdot (p_3 - p_1) \quad d_2 = (p_1 - p_2) \cdot (p_3 - p_2) \quad d_3 = (p_1 - p_3) \cdot (p_2 - p_3)
\]

\[
D = 2(d_1 d_2 + d_2 d_3 + d_3 d_1)
\]

Radius of circumcircle:

\[
R = \frac{1}{2} \sqrt{\frac{(d_1 + d_2)(d_2 + d_3)(d_3 + d_1)}{D/2}}
\]

Circumcenter can be far away from the vertices
⇒ In general not suited for practical use

Triangle \textit{not} affinely related to its circumcenter
Some Special Points

**Example:** Given triangle vertices

\[
p_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Edge lengths: \( s_1 = \sqrt{2} \), \( s_2 = 1 \), \( s_3 = 1 \)

Circumference of triangle: \( c = 2 + \sqrt{2} \)

The incenter:

\[
i \approx \left( \frac{\sqrt{2}}{2 + \sqrt{2}}, \frac{1}{2 + \sqrt{2}}, \frac{1}{2 + \sqrt{2}} \right) \approx (0.41, 0.29, 0.29)
\]

The coordinates of the incenter

\[
i = 0.41 \times p_1 + 0.29 \times p_2 + 0.29 \times p_3 = \begin{bmatrix} 0.29 \\ 0.29 \end{bmatrix}
\]

The circumcenter: \( d_1 = 0 \), \( d_2 = 1 \), \( d_3 = 1 \), \( D = 2 \)

\[
c \cong (0, 1/2, 1/2) \Rightarrow \text{midpoint of the “diagonal” edge}
\]

Radius of the circumcircle: \( R = \sqrt{2}/2 \)
2D Triangulations

Used by many applications e.g.,
— For centuries in surveying
— Finite element analysis

**Definition:** A set of triangles formed from a 2D points \( \{ p_i \}_{i=1}^N \) such that:

1. Vertices of the triangles consist of the \( p_i \);
2. Interiors of any two triangles do not intersect
3. If two triangles are not disjoint then they share a vertex or edge
4. Union of all triangles equals the convex hull of the \( p_i \)
2D Triangulations

Examples of \textit{illegal} triangulations

Top: overlapping triangles
Middle: boundary not the convex hull of points
Bottom: violates condition 3

Terminology:
\textbf{Valence}: number of triangles surrounding a vertex
\textbf{Star} triangles around a vertex
Non-uniqueness of triangulations

If we are given a point set, is there a unique triangulation?

Among the many possible triangulations the Delaunay triangulation commonly agreed to be the “best”
A Data Structure

Best data structure?
— storage requirements
— accessibility

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th>(number of points)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.0</td>
<td>0.0</td>
<td>(point 1)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.3</td>
<td>(\text{number of triangles})</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
<td>(1st triangle)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Important: consistent triangle orientation
An improved data structure: include *neighbor information*

5
0.0 0.0
1.0 0.0
0.0 1.0
0.25 0.3
0.5 0.3
5
1 2 5 2 4 -1
2 3 5 3 1 -1
4 5 3 2 5 4
1 5 4 3 5 1
1 4 3 3 -1 4

Triangle 1: points 1 2 5
— Across from point 1 is triangle 2
— Across from point 2 is triangle 4
— Across from point 5 is no triangle
Point location problem:
Given: point $p$ in the convex hull of the triangulation
Which triangle is $p$ in?

**Method 1:** Compute $p$’s barycentric coordinates with respect to all triangles — simple but expensive

**Method 2:** Use sign of barycentric coordinates to traverse triangulation

*Key:* If $p$ not in “current” triangle then move to neighboring triangle corresponding to a negative barycentric coordinate
Application: Point Location

Point Location Algorithm

1. Choose a guess triangle to be the current triangle $T$
2. Compute $p$’s barycentric coordinates $(u, v, w)$ with respect to $T$
3. If all barycentric coordinates are positive then output current triangle — exit
4. Determine the most negative of $(u, v, w)$
5. Set the current triangle to be the neighbor associated with this coordinate
6. Go to step 2

Can improve speed by not completing the division for determining the barycentric coordinates — must modify triangle inclusion test

If algorithm executed for more than one point can use previous run triangle as guess triangle — Take advantage of coherence in data set
3D Triangulations

• Triangles are connected to describe 3D geometric objects
• Rules for 3D triangulations same as for 2D
  — Data structure just adds $z$-coordinate in point list
• Shading requires a normal for each triangle or vertex
  — Normal is perpendicular to object’s surface at a particular point
  — Used to calculate how light is reflected \(\Rightarrow\) illumination of the object
barycentric coordinates
triangle inclusion test
affine invariance of barycentric coordinates
centroid, barycenter
incenter
circumcenter
2D triangulation criteria
star
valence
Delaunay triangulation
triangulation data structure
point location algorithm
3D triangulation criteria
3D triangulation data structure
normal