Practical Linear Algebra: A Geometry Toolbox
Fourth Edition

Chapter 17: Breaking It Up: Triangles

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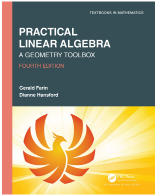
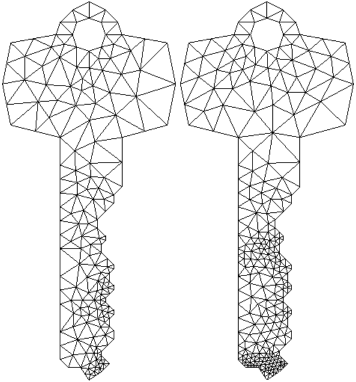


Outline

Introduction to Breaking It Up: Triangles

2D finite element method: refinement of a triangulation based on stress and strain calculations



Triangles are as old as geometry
Of interest to the ancient Greeks

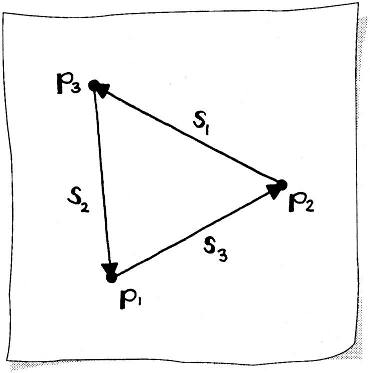
An indispensable tool in many applications

— computer graphics

— finite element analysis

Reducing the geometry to
linear or piecewise planar
makes computations more tractable

Barycentric Coordinates



A triangle T is given by three points

— Its vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$

— Vertices may live in 2D or 3D

Three points define a plane

$\Rightarrow$ a triangle is a 2D element

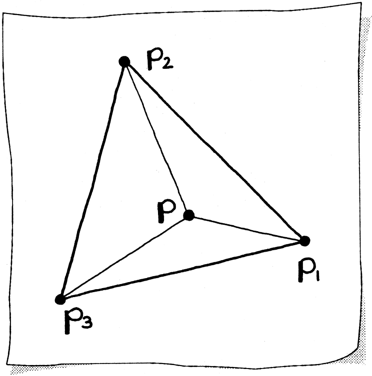
Conventions:

— Label the $\mathbf{p}_i$ counterclockwise

— Edge opposite point $\mathbf{p}_i$ labeled $\mathbf{s}_i$

Barycentric Coordinates

Invented by F. Moebius in 1827



Create a local coordinate system

Let $\mathbf{p}$ be an arbitrary point inside T

$$\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$$

Right-hand side:

a combination of points

$\Rightarrow$ coefficients must sum to one:

$$u + v + w = 1$$

As a linear system:

$$\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix}$$

Barycentric Coordinates

Solve the 3×3 linear system with *Cramer's rule*

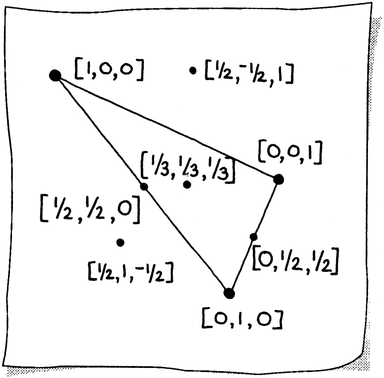
$$u = \frac{\text{area}(\mathbf{p}, \mathbf{p}_2, \mathbf{p}_3)}{\text{area}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)} \quad v = \frac{\text{area}(\mathbf{p}, \mathbf{p}_3, \mathbf{p}_1)}{\text{area}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)} \quad w = \frac{\text{area}(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2)}{\text{area}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$$

$\mathbf{u} = (u, v, w)$ called barycentric coordinates
of $\mathbf{p}$ with respect to $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, resp.

- Ratios of areas
- (u, v, w) sum to one $\Rightarrow$ not independent $w = 1 - u - v$
- Let $\mathbf{p} = \mathbf{p}_2 \Rightarrow v = 1$ and $u = w = 0$
- If $\mathbf{p}$ is on $\mathbf{s}_1$ then $u = 0$

Barycentric Coordinates

Examples of barycentric coordinates



Triangle vertices:

$$\mathbf{p}_1 \cong (1,0,0)$$
$$\mathbf{p}_2 \cong (0,1,0)$$
$$\mathbf{p}_3 \cong (0,0,1)$$

Even points *outside* of T have barycentric coordinates!

— Determinants return *signed* areas

Points inside T: positive (u, v, w)

Points outside T: mixed signs

Barycentric Coordinates

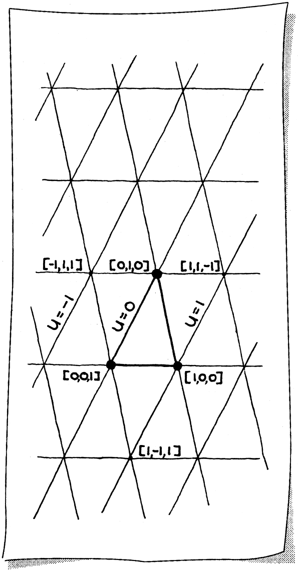
Application: Triangle inclusion test

Problem: Given a triangle T and a point $\mathbf{p}$. Is $\mathbf{p}$ is inside T?

Solution: Compute $\mathbf{p}$'s barycentric coordinates and check their signs!
— All the same sign then $\mathbf{p}$ is inside T
— Else $\mathbf{p}$ is outside T

Theoretically: one or two (u, v, w) could be zero $\Rightarrow$ $\mathbf{p}$ is on an edge

Numerically: not likely to encounter *exactly* zero
$\Rightarrow$ Do not test for equality
Instead: use a *zero tolerance* ϵ
Is $|\text{barycentric coordinate}| < \epsilon$?



Whole plane covered by a grid of coordinate lines

Plane divided into seven regions by the (extended) edges of T

Barycentric Coordinates

Example: Triangle vertices

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Points $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ with barycentric coordinates

$$\mathbf{q} \cong \left(0, \frac{1}{2}, \frac{1}{2}\right) \quad \mathbf{r} \cong (-1, 1, 1) \quad \mathbf{s} \cong \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

have coordinates in the plane

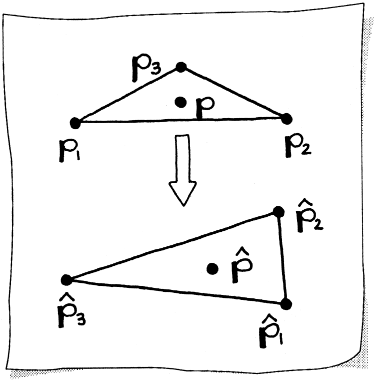
$$\mathbf{q} = 0 \times \mathbf{p}_1 + \frac{1}{2} \times \mathbf{p}_2 + \frac{1}{2} \times \mathbf{p}_3 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{r} = -1 \times \mathbf{p}_1 + 1 \times \mathbf{p}_2 + 1 \times \mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{s} = \frac{1}{3} \times \mathbf{p}_1 + \frac{1}{3} \times \mathbf{p}_2 + \frac{1}{3} \times \mathbf{p}_3 = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

Affine Invariance

Barycentric coordinates are affinely invariant



- $\hat{T}$ is an affine image of T
- $\mathbf{p} \cong \mathbf{u}$ relative to T
- $\hat{\mathbf{p}}$ is an affine image of $\mathbf{p}$

What are the barycentric coordinates of $\hat{\mathbf{p}}$ with respect to $\hat{T}$?

Ratios of areas are invariant under affine maps

— Individual areas change but not the ratios

$$\Rightarrow \hat{\mathbf{p}} \cong \mathbf{u} \text{ relative to } \hat{T}$$

Affine Invariance

Example: Given triangle vertices

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Apply a 90° rotation

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{Resulting in } \hat{\mathbf{p}}_i = R\mathbf{p}_i$$

Barycentric coordinates $(1/3, 1/3, 1/3)$ relative to T

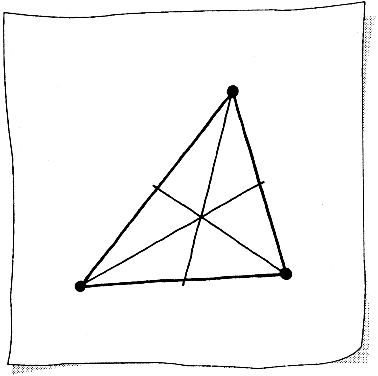
$$\mathbf{s} = \frac{1}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2 + \frac{1}{3}\mathbf{p}_3 = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathbf{s}} = R\mathbf{s} = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix}$$

Due to the affine invariance of barycentric coordinates could have found the coordinates as

$$\hat{\mathbf{s}} = \frac{1}{3}\hat{\mathbf{p}}_1 + \frac{1}{3}\hat{\mathbf{p}}_2 + \frac{1}{3}\hat{\mathbf{p}}_3 = \begin{bmatrix} -1/3 \\ 1/3 \end{bmatrix}$$

Some Special Points

The centroid $\mathbf{c}$



Intersection of the three medians

$$\mathbf{c} \cong \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Verify by writing

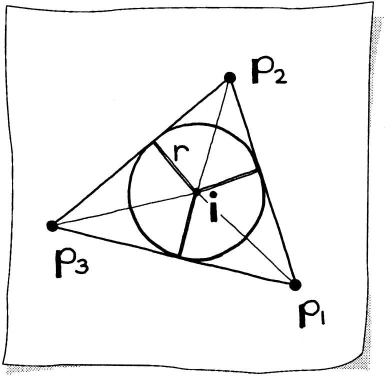
$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}(0,1,0) + \frac{2}{3}\left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

$\Rightarrow$ centroid lies on median associated with $\mathbf{p}_2$
— Same idea for remaining medians

Triangle is affinely related to its centroid

Some Special Points

The incenter $\mathbf{i} = (i_1, i_2, i_3)$



Intersection of the angle bisectors
$\mathbf{i}$ is the center of the *incircle*

s_i: length of triangle edge opposite $\mathbf{p}_i$
r: radius of the incircle

$$i_1 = \frac{\text{area}(\mathbf{i}, \mathbf{p}_2, \mathbf{p}_3)}{\text{area}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}$$

Use "1/2 base times height" rule

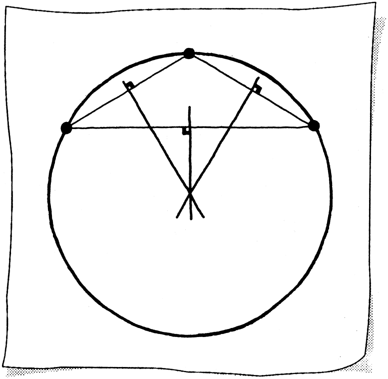
$$i_1 = \frac{rs_1}{rs_1 + rs_2 + rs_3}$$

$$i_1 = s_1/c \quad i_2 = s_2/c \quad i_3 = s_3/c$$

$c = s_1 + s_2 + s_3$ is circumference of T

Triangle is *not* affinely related to its incenter

The circumcenter **cc**



Circle through T's vertices called the circumcircle

Center of the circumcircle is the circumcenter

— Intersection of the edge bisectors

— Might not be inside the triangle

Some Special Points

The barycentric coordinates (cc_1, cc_2, cc_3) of the circumcenter

$$cc_1 = \frac{d_1(d_2 + d_3)}{D} \quad cc_2 = \frac{d_2(d_1 + d_3)}{D} \quad cc_3 = \frac{d_3(d_1 + d_2)}{D}$$

$$d_1 = (\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p}_3 - \mathbf{p}_1) \quad d_2 = (\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{p}_3 - \mathbf{p}_2) \quad d_3 = (\mathbf{p}_1 - \mathbf{p}_3) \cdot (\mathbf{p}_2 - \mathbf{p}_3)$$

$$D = 2(d_1 d_2 + d_2 d_3 + d_3 d_1)$$

Radius of circumcircle:

$$R = \frac{1}{2}\sqrt{\frac{(d_1 + d_2)(d_2 + d_3)(d_3 + d_1)}{D/2}}$$

Circumcenter can be far away from the vertices
$\Rightarrow$ In general not suited for practical use

Triangle *not* affinely related to its circumcenter

Some Special Points

Example: Given triangle vertices

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Edge lengths: $s_1 = \sqrt{2},\ s_2 = 1,\ s_3 = 1$
Circumference of triangle: $c = 2 + \sqrt{2}$

The incenter:

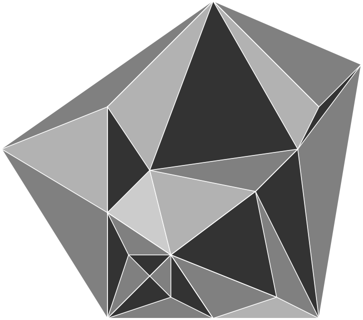
$$\mathbf{i} \cong \left(\frac{\sqrt{2}}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}}, \frac{1}{2+\sqrt{2}} \right) \approx (0.41, 0.29, 0.29)$$

The coordinates of the incenter

$$\mathbf{i} = 0.41 \times \mathbf{p}_1 + 0.29 \times \mathbf{p}_2 + 0.29 \times \mathbf{p}_3 = \begin{bmatrix} 0.29 \\ 0.29 \end{bmatrix}$$

The circumcenter: $d_1 = 0,\ d_2 = 1,\ d_3 = 1,\ D = 2$
$\mathbf{c} \cong (0, 1/2, 1/2) \Rightarrow$ midpoint of the "diagonal" edge
Radius of the circumcircle: $R = \sqrt{2}/2$

2D Triangulations

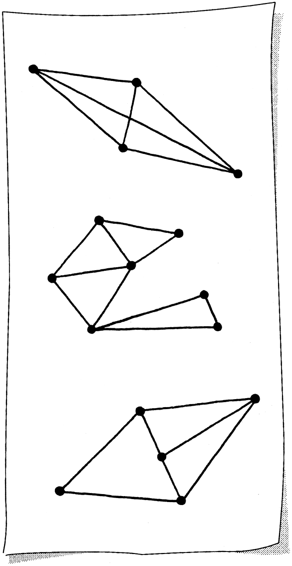


Used by many applications e.g.,
— For centuries in surveying
— Finite element analysis

Definition: A set of triangles formed from a 2D points $\{\mathbf{p}_i\}_{i=1}^{N}$ such that:

1. Vertices of the triangles consist of the $\mathbf{p}_i$
2. Interiors of any two triangles do not intersect
3. If two triangles are not disjoint then they share a vertex or edge
4. Union of all triangles equals the convex hull of the $\mathbf{p}_i$

2D Triangulations



Examples of *illegal* triangulations

Top: overlapping triangles

Middle: boundary not the convex hull of points

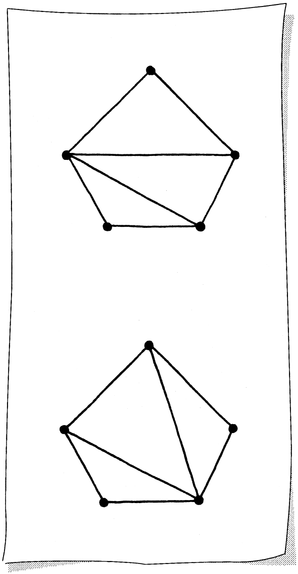
Bottom: violates condition 3

Terminology:

Valence: number of triangles surrounding a vertex

Star triangles around a vertex

2D Triangulations

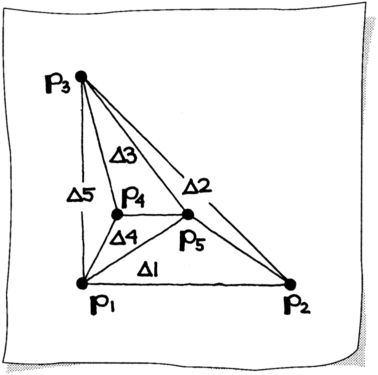


Non-uniqueness of triangulations

If we are given a point set, is there a unique triangulation?

Among the many possible triangulations the *Delaunay triangulation* commonly agreed to be the "best"

A Data Structure



Best data structure?

— storage requirements

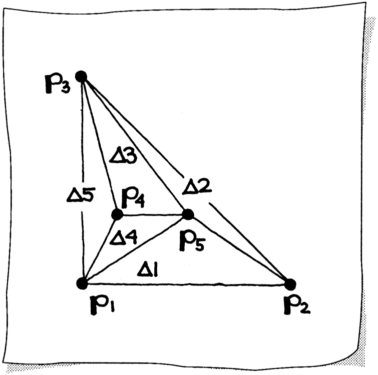
— accessibility

5			(number of points)
0.0	0.0		(point 1)
1.0	0.0		
0.0	1.0		
0.25	0.3		
0.5	0.3		
5			(number of triangles)
1	2	5	(1st triangle)
2	3	5	
4	5	3	
1	5	4	
1	4	3	

Important: consistent triangle orientation

A Data Structure

An improved data structure: include *neighbor information*



5					
0.0	0.0				
1.0	0.0				
0.0	1.0				
0.25	0.3				
0.5	0.3				
5					
1	2	5	2	4	-1
2	3	5	3	1	-1
4	5	3	2	5	4
1	5	4	3	5	1
1	4	3	3	-1	4

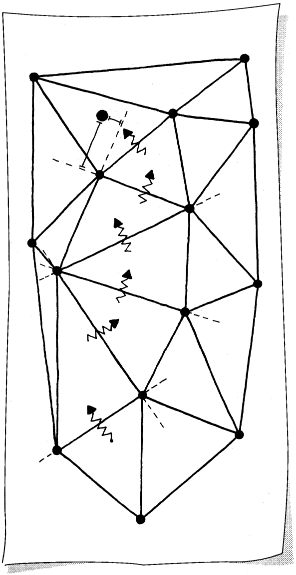
Triangle 1: points 1 2 5

— Across from point 1 is triangle 2

— Across from point 2 is triangle 4

— Across from point 5 is no triangle

Application: Point Location



Point location problem:
Given: point **p** in the convex hull of the triangulation
Which triangle is **p** in?

Method 1: Compute **p**'s barycentric coordinates with respect to all triangles — simple but expensive

Method 2: Use sign of barycentric coordinates to traverse triangulation

Key: If **p** not in "current" triangle then move to neighboring triangle corresponding to a negative barycentric coordinate

Application: Point Location

Point Location Algorithm

1. Choose a guess triangle to be the current triangle T
2. Compute $\mathbf{p}$'s barycentric coordinates (u, v, w) with respect to T
3. If all barycentric coordinates are positive then output current triangle — exit
4. Determine the most negative of (u, v, w)
5. Set the current triangle to be the neighbor associated with this coordinate
6. Go to step 2

Can improve speed by not completing the division for determining the barycentric coordinates — must modify triangle inclusion test

If algorithm executed for more than one point can use previous run triangle as guess triangle
— Take advantage of *coherence* in data set

3D Triangulations

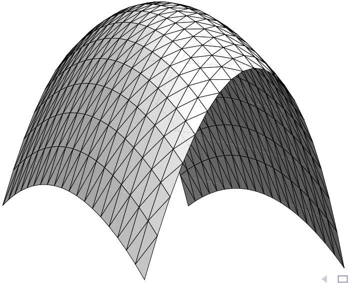
- Triangles are connected to describe 3D geometric objects
- Rules for 3D triangulations same as for 2D

— Data structure just adds z-coordinate in point list

- Shading requires a *normal* for each triangle or vertex

— Normal is perpendicular to object's surface at a particular point

— Used to calculate how light is reflected $\Rightarrow$ illumination of the object



WYSK

- barycentric coordinates
- triangle inclusion test
- affine invariance of barycentric coordinates
- centroid, barycenter
- incenter
- circumcenter
- 2D triangulation criteria
- star
- valence
- Delaunay triangulation
- triangulation data structure
- point location algorithm
- 3D triangulation criteria
- 3D triangulation data structure
- normal