Practical Linear Algebra: A GEOMETRY TOOLBOX
Fourth Edition

Chapter 20: Curves

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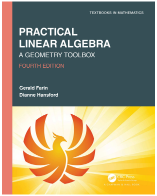
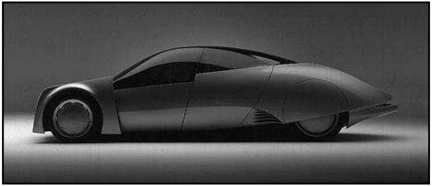


Outline

Introduction to Curves



Focus of this chapter: cubic Bézier curves

Invented for car design
— France early 1960s at Rénault and Citroën in Paris
— Techniques still in use today
— called *Geometric Modeling* or *Computer-Aided Geometric Design*

Apply linear algebra and geometric concepts to the study of curves

Parametric Curves

Straight line in *parametric* form:

$$\mathbf{x}(t) = (1-t)\mathbf{a} + t\mathbf{b} \quad \Rightarrow \quad \mathbf{x}(0) = \mathbf{a} \text{ and } \quad \mathbf{x}(1) = \mathbf{b}$$

Interpret t as time and $\mathbf{x}(t)$ as a moving point
Coefficients $(1-t)$ and t are linear polynomials $\Rightarrow$ *linear interpolation*

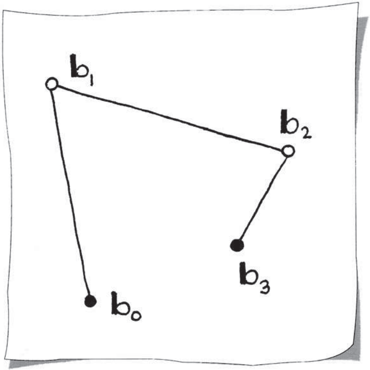
A parametric curve: a curve that can be written as

$$\mathbf{x}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

where $f(t)$ and $g(t)$ are functions of the parameter t

Linear interpolant: $f(t) = (1-t)a_1 + tb_1$ and $g(t) = (1-t)a_2 + tb_2$
In general: f and g can be any functions
— polynomial, trig, exponential, ...

Parametric Curves



Next: study motion along *curves*

Cubic Bézier curves:
Start with four points in 2D or 3D

$$\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$$

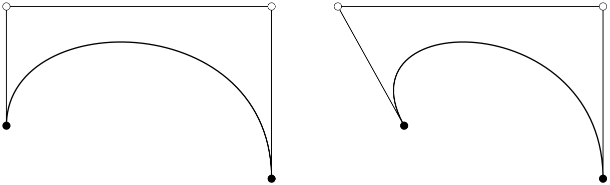
Bézier control points
⇒ Bézier controls polygon
— not assumed to be closed

Parametric Curves

To create plots: evaluate the cubic curve at many t-parameters
— $t \in [0, 1]$
— Example: evaluate at 50 points

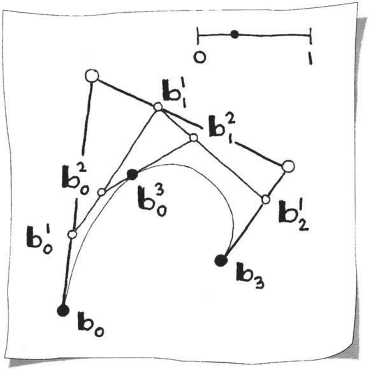
$$t = 0, 1/50, 2/50, \ldots, 49/50, 1$$

Evaluation points connected by straight line segments
Choose enough t-parameters so the curve looks smooth
$\Rightarrow$ plotting a *discrete approximation* of the curve



Two examples that differ in the location of $\mathbf{b}_0$ only

Evaluation: the de Casteljau algorithm



Generate one point
— Pick a parameter value $t \in [0, 1]$
— Linear interpolation on each leg

$$\mathbf{b}_0^1(t) = (1 - t)\mathbf{b}_0 + t\mathbf{b}_1$$
$$\mathbf{b}_1^1(t) = (1 - t)\mathbf{b}_1 + t\mathbf{b}_2$$
$$\mathbf{b}_2^1(t) = (1 - t)\mathbf{b}_2 + t\mathbf{b}_3$$

Repeat on new polygon

$$\mathbf{b}_0^2 = (1 - t)\mathbf{b}_0^1(t) + t\mathbf{b}_1^1(t)$$
$$\mathbf{b}_1^2 = (1 - t)\mathbf{b}_1^1(t) + t\mathbf{b}_2^1(t)$$

Repeat

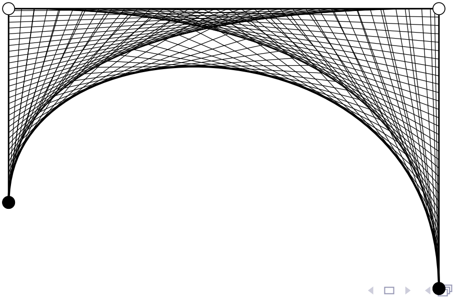
$$\mathbf{b}_0^3(t) = (1 - t)\mathbf{b}_0^2(t) + t\mathbf{b}_1^2(t)$$

$\Rightarrow$ Point on the Bézier curve

Points $\mathbf{b}_i^j$ are called intermediate Bézier points

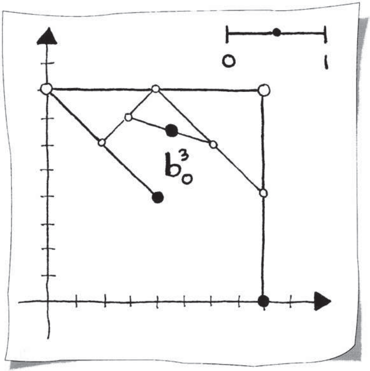
$$\text{Schematic} \quad \begin{array}{llll} \mathbf{b}_0 & & & \\ \mathbf{b}_1 & \mathbf{b}_0^1 & & \\ \mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\ \mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 \\ \text{stage}: & 1 & 2 & 3 \end{array}$$

Except for (input) Bézier polygon each point is a function of t



Example: de Casteljau algorithm at $t = 1/2$

$$\mathbf{b}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 8 \\ 8 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$



$$\mathbf{b}_0^1 = \frac{1}{2}\mathbf{b}_0 + \frac{1}{2}\mathbf{b}_1 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\mathbf{b}_1^1 = \frac{1}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\mathbf{b}_2^1 = \frac{1}{2}\mathbf{b}_2 + \frac{1}{2}\mathbf{b}_3 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\mathbf{b}_0^2 = \frac{1}{2}\mathbf{b}_0^1 + \frac{1}{2}\mathbf{b}_1^1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\mathbf{b}_1^2 = \frac{1}{2}\mathbf{b}_1^1 + \frac{1}{2}\mathbf{b}_2^1 = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\mathbf{b}_0^3 = \frac{1}{2}\mathbf{b}_0^2 + \frac{1}{2}\mathbf{b}_1^2 = \begin{bmatrix} \frac{9}{2} \\ \frac{13}{2} \end{bmatrix}$$

Equation for a cubic Bézier curve

$$\text{Expand:} \quad \mathbf{b}_0^3(t) = (1-t)\mathbf{b}_0^2(t) + t\mathbf{b}_1^2(t)$$

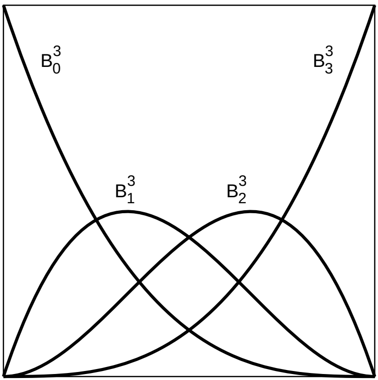
$$\begin{aligned}
\mathbf{b}_0^3 &= (1-t)\mathbf{b}_0^2 + t\mathbf{b}_1^2 \\
&= (1-t)\left[(1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1\right] + t\left[(1-t)\mathbf{b}_1^1 + t\mathbf{b}_2^1\right] \\
&= (1-t)\left[(1-t)\left[(1-t)\mathbf{b}_0 + t\mathbf{b}_1\right] + t\left[(1-t)\mathbf{b}_1 + t\mathbf{b}_2\right]\right] \\
&\quad + t\left[(1-t)\left[(1-t)\mathbf{b}_1 + t\mathbf{b}_2\right] + t\left[(1-t)\mathbf{b}_2 + t\mathbf{b}_3\right]\right]
\end{aligned}$$

Collect terms with the same $\mathbf{b}_i$

$$\mathbf{b}_0^3(t) = (1-t)^3\mathbf{b}_0 + 3(1-t)^2 t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}_3$$

General form of a cubic Bézier curve
$\mathbf{b}_0^3(t)$ traces out a curve as t traces out values between 0 and 1
Shorter notation: $\mathbf{b}(t)$ instead of $\mathbf{b}_0^3(t)$

Bernstein basis functions



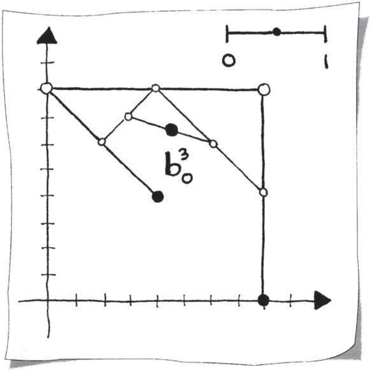
Cubic polynomials — degree 3

$$B_0^3(t) = (1-t)^3$$
$$B_1^3(t) = 3(1-t)^2 t$$
$$B_2^3(t) = 3(1-t)t^2$$
$$B_3^3(t) = t^3$$

$$\mathbf{b}_0^3(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

All cubic polynomials can be expressed in terms of Bernstein polynomials
$\Rightarrow$ form a basis for all cubic polynomials

Subdivision



A subset of intermediate Bézier points form two cubic polygons
— Mimic the curve's shape

Curve segment from $\mathbf{b}_0$ to $\mathbf{b}_0^3(t)$

$$\mathbf{b}_0, \mathbf{b}_0^1, \mathbf{b}_0^2, \mathbf{b}_0^3$$

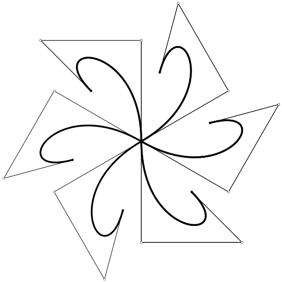
Curve segment from $\mathbf{b}_0^3(t)$ to $\mathbf{b}_3$

$$\mathbf{b}_0^3, \mathbf{b}_1^2, \mathbf{b}_2^1, \mathbf{b}_3$$

Identify polygons in schematic

$$\begin{array}{llll} \mathbf{b}_0 & & & \\ \mathbf{b}_1 & \mathbf{b}_0^1 & & \\ \mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\ \mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 \end{array}$$

Properties of Bézier Curves



Endpoint interpolation: $\mathbf{b}(0) = \mathbf{b}_0$ and $\mathbf{b}(1) = \mathbf{b}_3$

Affine invariance: affine map of the control polygon
then curve undergoes the same transformation
$\Rightarrow$ Bernstein polynomials sum to one

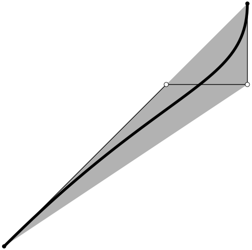
$$(1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 = [(1-t)+t]^3 = 1$$

Every point on the curve is a *barycentric combination* of the control points

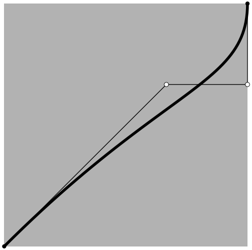
Properties of Bézier Curves

Bernstein polynomials are nonnegative for $t \in [0, 1]$
$\Rightarrow$ Every point on the curve is a *convex combination* of the control points
$\Rightarrow$ Convex hull property: for $t \in [0, 1]$ the curve lies in the convex hull of the control polygon

Extrapolation: evaluate the curve for t-values outside of $[0, 1]$
— Can no longer predict the shape of the curve



convex hull



minmax box

The convex hull lies inside the minmax box

The Matrix Form

Bézier curve expressed using dot products:

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} (1-t)^3 \\ 3(1-t)^2 t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix}$$

Most well-known: cubic polynomials as combinations of the monomials:

$$1, t, t^2, t^3$$

Rewrite the Bézier curve:

$$\mathbf{b}(t) = \mathbf{b}_0 + 3t(\mathbf{b}_1 - \mathbf{b}_0) + 3t^2(\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0) + t^3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0)$$

$\Rightarrow$ Matrix form of a Bézier curve

$$\mathbf{b}(t) = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The Matrix Form

A curve in monomial form:

$$\begin{aligned}\mathbf{b}(t) &= \mathbf{b}_0 + 3t(\mathbf{b}_1 - \mathbf{b}_0) + 3t^2(\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0) + t^3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0) \\ &= \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3\end{aligned}$$

Geometrically: $\mathbf{a}_0 = \mathbf{b}_0$ is a point $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are vectors

Using the dot product form:

$$\mathbf{b}(t) = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} \begin{bmatrix}1 \\ t \\ t^2 \\ t^3\end{bmatrix}$$

$\Rightarrow$ the monomial $\mathbf{a}_i$ are defined as

$$\begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} = \begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1\end{bmatrix}$$

The Matrix Form

The inverse process:
Given a curve in monomial form, how to write it as a Bézier curve?

$$\begin{bmatrix}\mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = \begin{bmatrix}\mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} \begin{bmatrix}1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1\end{bmatrix}^{-1}$$

A matrix inversion is all that is needed here!
— Matrix is nonsingular
$\Rightarrow$ Any cubic curve can be written in either Bézier or monomial form

Derivatives

Cubic Bézier curve: two (in 2D) or three (in 3D) cubic equations in t

$$\mathbf{b}_0^3(t) = (1-t)^3\mathbf{b}_0 + 3(1-t)^2 t\mathbf{b}_1 + 3(1-t)t^2\mathbf{b}_2 + t^3\mathbf{b}_3$$

First derivative vector: derivative in each of the components

$$\begin{aligned}\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} &= -3(1-t)^2\mathbf{b}_0 - 6(1-t)t\mathbf{b}_1 + 3(1-t)^2\mathbf{b}_1 \\ &\quad - 3t^2\mathbf{b}_2 + 6(1-t)t\mathbf{b}_2 + 3t^2\mathbf{b}_3\end{aligned}$$

Rearrange and use abbreviation $\frac{\mathrm{d}\mathbf{b}(t)}{\mathrm{d}t} = \dot{\mathbf{b}}(t)$

$$\dot{\mathbf{b}}(t) = 3(1-t)^2[\mathbf{b}_1 - \mathbf{b}_0] + 6(1-t)t[\mathbf{b}_2 - \mathbf{b}_1] + 3t^2[\mathbf{b}_3 - \mathbf{b}_2]$$

$\Rightarrow$ Derivative curve is degree two
$\Rightarrow$ Derivative curve has *control vectors*

Derivatives

One very nice feature of the de Casteljau algorithm:
intermediate Bézier points lead to simple expression of the derivative

$$\dot{\mathbf{b}}(t) = 3\left[\mathbf{b}_1^2(t) - \mathbf{b}_0^2(t)\right]$$

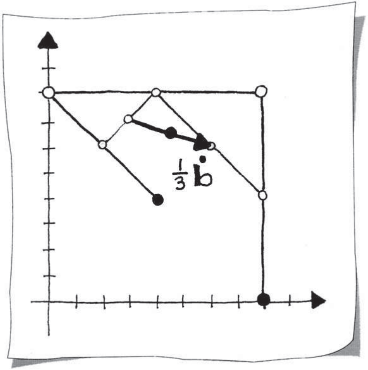
Very simple form at the endpoints

$$\dot{\mathbf{b}}(0) = 3[\mathbf{b}_1 - \mathbf{b}_0] \qquad \dot{\mathbf{b}}(1) = 3[\mathbf{b}_3 - \mathbf{b}_2]$$

Control polygon is tangent to the curve at the curve's endpoints

Derivatives

Example: Compute first derivative vector for $t = 1/2$



Evaluate the direct equation

$$\dot{\mathbf{b}}\left(\frac{1}{2}\right) = 3 \cdot \frac{1}{4}\left[\begin{bmatrix}0\\8\end{bmatrix} - \begin{bmatrix}4\\4\end{bmatrix}\right] + 6 \cdot \frac{1}{4}\left[\begin{bmatrix}8\\8\end{bmatrix} - \begin{bmatrix}0\\8\end{bmatrix}\right] + 3 \cdot \frac{1}{4}\left[\begin{bmatrix}8\\0\end{bmatrix} - \begin{bmatrix}8\\8\end{bmatrix}\right] = \begin{bmatrix}9\\-3\end{bmatrix}$$

Use intermediate control points

$$\dot{\mathbf{b}}\left(\frac{1}{2}\right) = 3\left[\mathbf{b}_1^2\left(\frac{1}{2}\right) - \mathbf{b}_0^2\left(\frac{1}{2}\right)\right] = 3\left[\begin{bmatrix}6\\6\end{bmatrix} - \begin{bmatrix}3\\7\end{bmatrix}\right] = \begin{bmatrix}9\\-3\end{bmatrix}$$

Derivatives

The derivative of a curve is a *vector*
— It is tangent to the curve

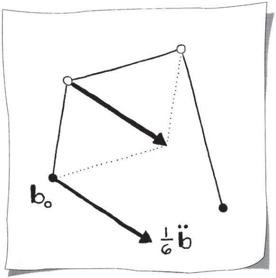
Can interpret it as a *velocity vector*
— Interpret parameter t as time
then arrive at $\mathbf{b}(t)$ at time t with velocity $\dot{\mathbf{b}}(t)$
— Large magnitude of the tangent vector $\Rightarrow$ moving fast

If we rotate the control polygon, the curve will follow
— And so will all of its derivative vectors

In calculus: a "horizontal tangent" has a special meaning
— It indicates an extreme value of a function
— Notion of an extreme value is meaningless for parametric curves
- ◇ Term "horizontal tangent" depends on the curve's orientation
- ◇ Not a property of the curve itself

Derivatives

Second derivative vector: may interpret as acceleration



$$\dot{\mathbf{b}}(t) = 3(1-t)^2[\mathbf{b}_1 - \mathbf{b}_0] + 6(1-t)t[\mathbf{b}_2 - \mathbf{b}_1] + 3t^2[\mathbf{b}_3 - \mathbf{b}_2]$$

$$\ddot{\mathbf{b}}(t) = -6(1-t)[\mathbf{b}_1 - \mathbf{b}_0] - 6t[\mathbf{b}_2 - \mathbf{b}_1] + 6(1-t)[\mathbf{b}_2 - \mathbf{b}_1] + 6t[\mathbf{b}_3 - \mathbf{b}_2]$$
$$= 6(1-t)[\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0] + 6t[\mathbf{b}_3 - 2\mathbf{b}_2 + \mathbf{b}_1]$$

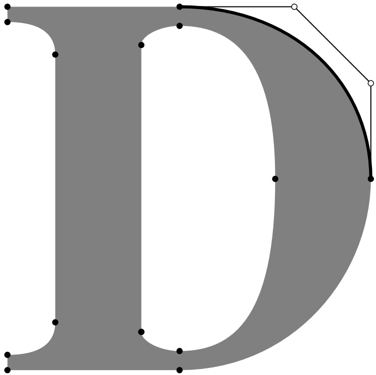
Via the de Casteljau algorithm:

$$\ddot{\mathbf{b}}(t) = 6\left[\mathbf{b}_2^1(t) - 2\mathbf{b}_1^1(t) + \mathbf{b}_0^1(t)\right]$$

Simple form at endpoints: $\ddot{\mathbf{b}}(0) = 6[\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0]$

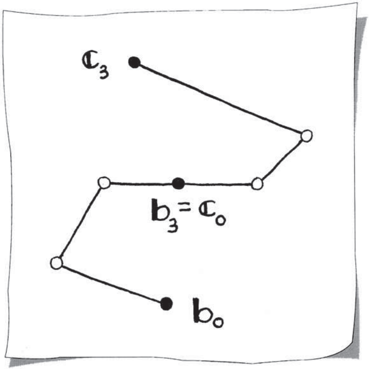
Composite Curves

For "real" shapes need to use many cubic Bézier curves
— Smooth overall curve ⇒ pieces must join smoothly



The letter "**D**" as a collection of cubic Bézier curves
— Only one Bézier polygon of many is shown

Smoothly joining Bézier curves



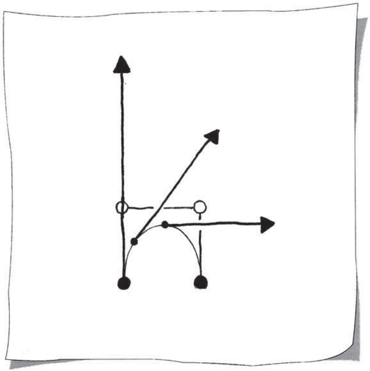
Two Bézier curves:
$\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$
Common point $\mathbf{b}_3 = \mathbf{c}_0$

Same tangent vector direction:

$$\mathbf{c}_1 - \mathbf{c}_0 = c[\mathbf{b}_3 - \mathbf{b}_2]$$

For some positive real number c
$\Rightarrow \mathbf{b}_2,\ \mathbf{b}_3 = \mathbf{c}_0,\ \mathbf{c}_1$ collinear

The Geometry of Planar Curves



Imagine driving with constant speed
Curviness of road proportional to turning of steering wheel

Sample tangents at various points
— Successive tangents differ significantly
$\Rightarrow$ curvature is high
— Successive tangents are almost identical
$\Rightarrow$ curvature is low

Curvature: rate of change of tangents

The Geometry of Planar Curves

Tangent determined by the first derivative
⇒ its rate of change should also involve the second derivative

$$\text{Curvature:} \qquad \kappa(t) = \frac{\|\dot{\mathbf{b}} \wedge \ddot{\mathbf{b}}\|}{\|\dot{\mathbf{b}}\|^3}$$

2D: can use special formula

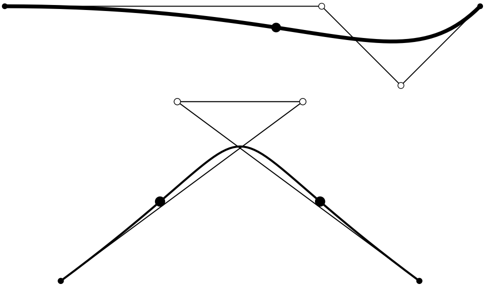
$$\kappa(t) = \frac{|\dot{\mathbf{b}} \quad \ddot{\mathbf{b}}|}{\|\dot{\mathbf{b}}\|^3}$$

⇒ 2D curvature is *signed*

Inflection point: a point where $\kappa = 0$

— Curvature changes sign on either side of the inflection point

— First and second derivative vectors are parallel or linearly dependent

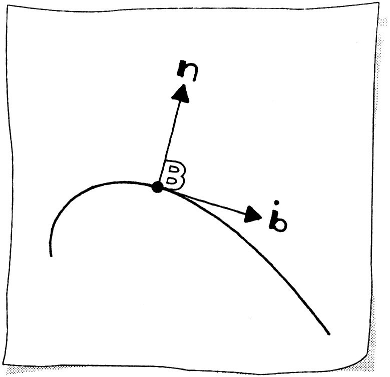


Moving along a Curve

Curve motions: "B" sliding along a Bézier curve

Moving along a Curve

Position an object at a point on a 2D curve



Point on curve: $\mathbf{b}(t)$

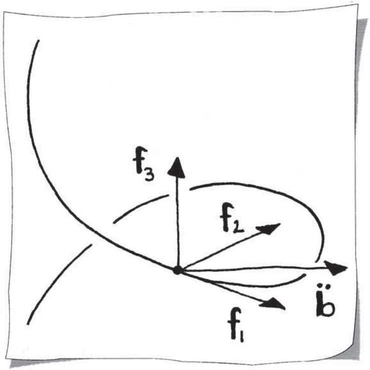
Tangent $\dot{\mathbf{b}}$

Vector $\mathbf{n}$ perpendicular to tangent

$\Rightarrow$ Local coordinate system

Moving along a Curve

Position an object at a point on a 3D curve



Frenet frame

$$\mathbf{f}_1 = \frac{\dot{\mathbf{b}}(t)}{\|\dot{\mathbf{b}}(t)\|}$$

$$\mathbf{f}_3 = \frac{\dot{\mathbf{b}}(t) \wedge \ddot{\mathbf{b}}(t)}{\|\dot{\mathbf{b}}(t) \wedge \ddot{\mathbf{b}}(t)\|}$$

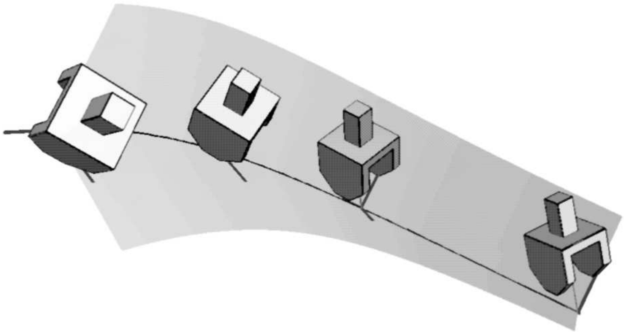
$$\mathbf{f}_2 = \mathbf{f}_3 \wedge \mathbf{f}_1$$

Osculating plane:
defined by $\dot{\mathbf{b}}(t)$ and $\ddot{\mathbf{b}}(t)$

Moving along a Curve

Suppose an object is in some local coordinate system with axes $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$
Any point of the object has coordinates $\mathbf{u}$

$$\text{Mapped to:} \quad \mathbf{x}(t, \mathbf{u}) = \mathbf{b}(t) + u_1\mathbf{f}_1 + u_2\mathbf{f}_2 + u_3\mathbf{f}_3$$



A typical application: *robot motion*
— A robot arm is moved along a curve

WYSK

- linear interpolation
- parametric curve
- de Casteljau algorithm
- cubic Bézier curve
- subdivision
- affine invariance
- convex hull property
- Bernstein polynomials
- basis function
- barycentric combination
- matrix form
- cubic monomial curve
- Bernstein and monomial conversion
- nonsingular
- first derivative
- second derivative
- parallelogram rule
- composite Bézier curves
- curvature
- inflection point
- Frenet frame
- osculating plane