Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 5: 2×2 Linear Systems

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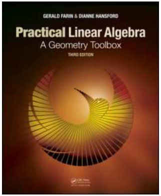
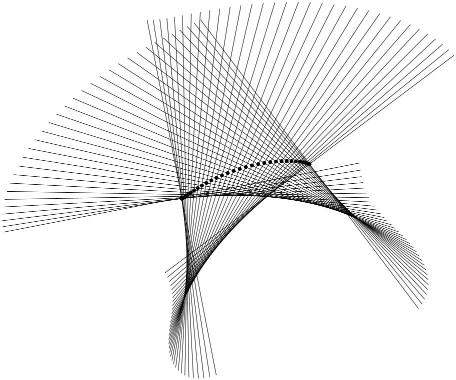


Outline

Introduction to 2 × 2 Linear Systems



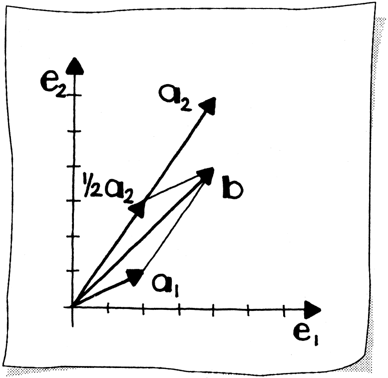
Two families of lines are shown

Intersections of corresponding line pairs marked

For each intersection:
solve a 2 × 2 linear system

Skew Target Boxes Revisited

Geometry of a 2 $\times$ 2 system



$\mathbf{a}_1$ and $\mathbf{a}_2$ define a skew target box

Given $\mathbf{b}$ with respect to the $[\mathbf{e}_1, \mathbf{e}_2]$-system:
What are the components of $\mathbf{b}$ with respect to the $[\mathbf{a}_1, \mathbf{a}_2]$-system?

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$1 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Skew Target Boxes Revisited

Two equations in the two unknowns u_1 and u_2

$$2u_1 + 4u_2 = 4$$
$$u_1 + 6u_2 = 4$$

Solution: $u_1 = 1$ and $u_2 = 1/2$

This chapter dedicated to solving these equations

The Matrix Form

Two equations

$$a_{1,1}u_1 + a_{1,2}u_2 = b_1$$
$$a_{2,1}u_1 + a_{2,2}u_2 = b_2$$

Also called a linear system

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A\mathbf{u} = \mathbf{b}$$

$\mathbf{u}$ called the solution of linear system

Previous example:

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

The Matrix Form

Recall geometric interpretation of $A\mathbf{u} = \mathbf{b}$:
Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$

$$u_1\mathbf{a}_1 + u_2\mathbf{a}_2 = \mathbf{b}$$

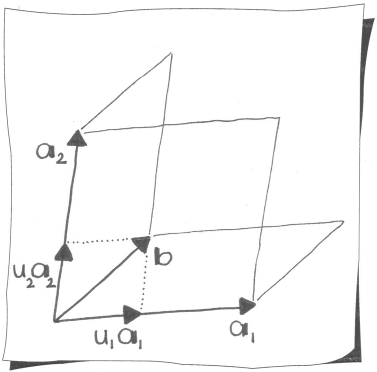
At least one solution: linear system called consistent
Otherwise: called inconsistent

Three possibilities for solution space:

1. Exactly one solution vector $\mathbf{u}$
   $|A| \neq 0$
   matrix has full rank and is non-singular
2. No solution (system is inconsistent)
3. Infinitely many solutions

(Sketches of each case to come)

A Direct Approach: Cramer's Rule



$$u_1 = \frac{\text{area}(\mathbf{b}, \mathbf{a}_2)}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)} \qquad u_2 = \frac{\text{area}(\mathbf{a}_1, \mathbf{b})}{\text{area}(\mathbf{a}_1, \mathbf{a}_2)}$$

Ratios of areas

Shear parallelogram formed by
— $\mathbf{b}, \mathbf{a}_2$ onto $\mathbf{a}_1$
— $\mathbf{b}, \mathbf{a}_1$ onto $\mathbf{a}_2$
(Shears preserve areas)

Signed area of a parallelogram given by determinant

A Direct Approach: Cramer's Rule

Example:

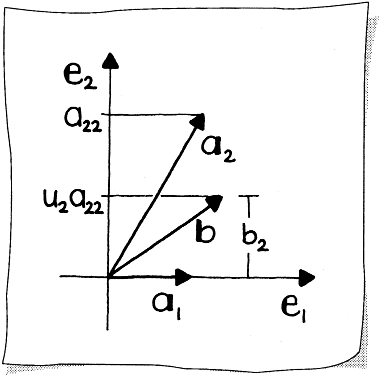
$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$u_1 = \frac{\begin{vmatrix} 4 & 4 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{8}{8} \qquad u_2 = \frac{\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}} = \frac{4}{8}$$

What if area spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ is zero?

Cramer's rule primarily of theoretical importance
For larger systems: expensive and numerically unstable

Gauss Elimination



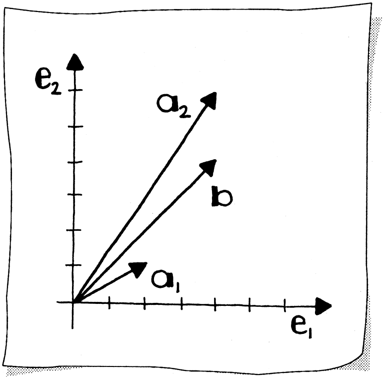
Special 2×2 linear system:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ 0 & a_{2,2} \end{bmatrix} \mathbf{u} = \mathbf{b}$$

Matrix is called *upper triangular*
Solve with back substitution:

$$u_2 = b_2 / a_{2,2}$$
$$u_1 = \frac{1}{a_{1,1}} (b_1 - u_2 a_{1,2})$$

Diagonal elements key: called pivots



Any linear system with non-singular matrix may be *transformed* to upper triangular via forward elimination

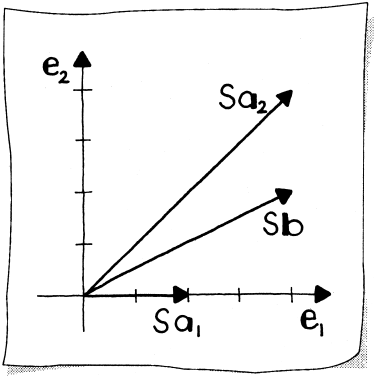
Process of forward elimination followed by back substitution is called *Gauss elimination*

Example:

$$u_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Key fact: *linear maps do not change linear combinations*

Gauss Elimination



Apply the same linear map to all vectors in system
— factors u_1 and u_2 won't change:

$$S\left[u_1\begin{bmatrix}2\\1\end{bmatrix}+u_2\begin{bmatrix}4\\6\end{bmatrix}\right]=S\begin{bmatrix}4\\4\end{bmatrix}$$

Shear parallel to the $\mathbf{e}_2$-axis so that

$$\begin{bmatrix}2\\1\end{bmatrix}\text{ is mapped to }\begin{bmatrix}2\\0\end{bmatrix}$$

Shear matrix:

$$S_1=\begin{bmatrix}1&0\\-1/2&1\end{bmatrix}$$

Gauss Elimination

Transformed system:

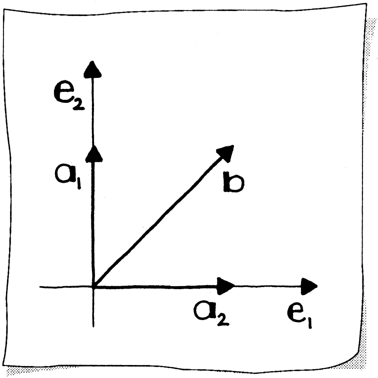
$$\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Next: back substitution

$$u_2 = 2/4 = 1/2,$$
$$u_1 = \frac{1}{2}\left(4 - 4 \times \frac{1}{2}\right) = 1$$

2×2 linear systems:
only one matrix entry to zero in the forward elimination procedure
More algorithmic approach in Chapter 12 Gauss for Linear Systems

Pivoting



Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Shearing $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis will not work

Solution: exchange two equations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Exchanging equations (rows) so pivot is the largest in absolute value called {row, partial} pivoting

Used to improve numerical stability

Pivoting

Example:

$$\begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Shear $\mathbf{a}_1$ onto the $\mathbf{e}_1$-axis

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9998 \end{bmatrix}$$

Performing back substitution

$$\mathbf{u}_t = \begin{bmatrix} 1.0001 \\ 0.9998\bar{9} \end{bmatrix} \qquad (\text{“true” solution})$$

Suppose machine only stores three digits — system stored as

$$\begin{bmatrix} 0.0001 & 1 \\ 0 & -10000 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10000 \end{bmatrix},$$

$$\mathbf{u}_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (\text{“round-off” solution})$$

Not very close to the true solution $\mathbf{u}_t$: $\|\mathbf{u}_t - \mathbf{u}_r\| = 1.0001$

Pivoting

Pivoting dampers effects of round-off

$$\begin{bmatrix} 1 & 1 \\ 0.0001 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Forward elimination

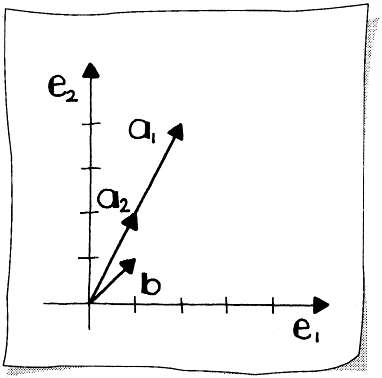
$$\begin{bmatrix} 1 & 1 \\ 0 & 0.9999 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.9998 \end{bmatrix}$$

$$\mathbf{u}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{(“pivoting” solution)}$$

Closer to “true” solution: $\|\mathbf{u}_t - \mathbf{u}_p\| = 0.00014$

Unsolvable Systems

$\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent



$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Forward elimination (shear $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis):

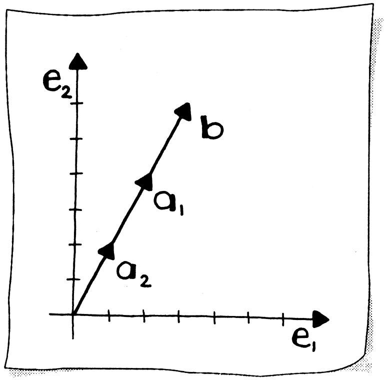
$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Last equation: $0 = -1$
System is *inconsistent* $\Rightarrow$ no solution

Approximate solution via least squares methods

Underdetermined Systems

$\mathbf{b}$ a multiple of $\mathbf{a}_1$ or $\mathbf{a}_2$



$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Forward elimination (shear $\mathbf{a}_1$ onto $\mathbf{e}_1$-axis):

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Last equation: $0 = 0$ — true, but a bit trivial

One equation written twice

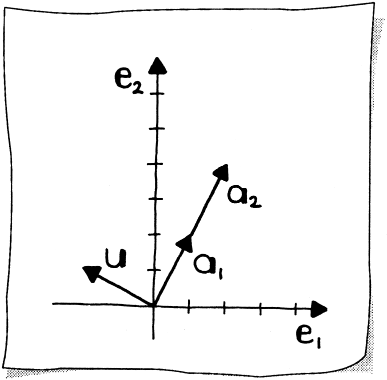
System *underdetermined*

System *consistent*: at least one solution exists

— example: set $u_2 = 1$ then $u_1 = 1$

Homogeneous Systems

$A\mathbf{u} = \mathbf{0}$



Trivial solution solution: $\mathbf{u} = \mathbf{0}$
— usually of little interest

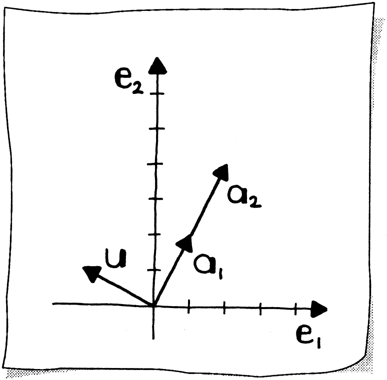
If solution $\mathbf{u} \neq \mathbf{0}$ exists
then all $c\mathbf{u}$ are solutions
$\Rightarrow$ *infinite number of solutions*

Not all homogeneous systems have a nontrivial solution
— only rank 1 2×2 maps
$\Rightarrow$ $\mathbf{a}_1$ and $\mathbf{a}_2$ linearly dependent

Only trivial solution $\Rightarrow$ A invertible

Homogeneous Systems

$A\mathbf{u} = \mathbf{0}$



$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1 \Rightarrow A$ maps all vectors onto line defined by $\mathbf{0}, \mathbf{a}_1$

Forward elimination:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Pick $u_2 = 1$ then

back substitution gives $u_1 = -2$

Any $c\mathbf{u}$ perpendicular to $\mathbf{a}_1$ a solution: $\mathbf{a}_1 \cdot \mathbf{u} = 0$

— make up kernel or null space of A

Homogeneous Systems

Example: only the trivial solution:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Columns of A are linearly independent
A does not reduce dimensionality
$\Rightarrow$ cannot map $\mathbf{u} \neq \mathbf{0}$ to $\mathbf{0}$
Forward elimination:

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Back substitution: $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Example: row pivoting not helpful – need column pivoting

$$\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} \mathbf{u} = \mathbf{0}.$$

Column pivoting:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \mathbf{0}.$$

(Exchange unknowns too)
Set $u_1 = 1$ and back substitution results in $u_2 = 0$
Solutions: $\mathbf{u} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Undoing Maps: Inverse Matrices

How to *undo* a linear map
Given $A\mathbf{u} = \mathbf{b}$
what matrix B maps $\mathbf{b}$ back to $\mathbf{u}$: $\mathbf{u} = B\mathbf{b}$? B is the *inverse map*

Recall: shears can be used to zero matrix elements

$$S_1 A\mathbf{u} = S_1\mathbf{b}.$$

Example:

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Undoing Maps: Inverse Matrices

Second shear: $S_2 S_1 A\mathbf{u} = S_2 S_1 \mathbf{b}$ (map new $\mathbf{a}_2$ to the $\mathbf{e}_2$-axis)

$$S_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{results in} \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Non-uniform scaling S_3 (map $\mathbf{a}_1$, $\mathbf{a}_2$ onto $\mathbf{e}_1$, $\mathbf{e}_2$)

$$S_3 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \quad \text{results in} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

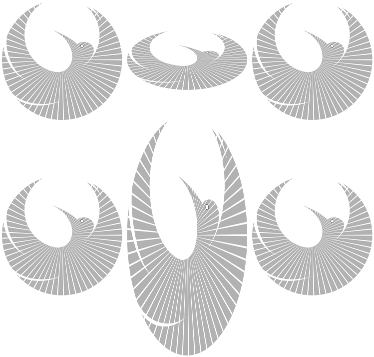
All together:

$$S_3 S_2 S_1 A\mathbf{u} = S_3 S_2 S_1 \mathbf{b}$$
$$I\mathbf{u} = A^{-1}\mathbf{b}$$

I called the identity matrix
A^{-1} called the inverse matrix and A called invertible

Undoing Maps: Inverse Matrices

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I \qquad\qquad I^{-1} = I$$



Inverse of a scaling:

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 0 \\ 0 & 1/t \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Top: original Phoenix, scale, inverse scale
Bottom: original Phoenix, inverse scale, original scale

Undoing Maps: Inverse Matrices

Top: original Phoenix, shear, inverse shear
Bottom: original Phoenix, inverse shear, original shear

Undoing Maps: Inverse Matrices

Rotation matrices:

$$R_{-\alpha} = R_{\alpha}^{-1} = R_{\alpha}^{\mathrm{T}}$$

Rotation matrix is an *orthogonal matrix*:

$$A^{-1} = A^{\mathrm{T}}$$

Column vectors satisfy $\|\mathbf{a}_1\| = 1$, $\|\mathbf{a}_2\| = 1$ and $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$
$\rightarrow$ vectors called orthonormal
$\Rightarrow$ these linear maps called *rigid body motions*
Characterized by determinant $= \pm 1$

Undoing Maps: Inverse Matrices

$$A^{-1^{-1}} = A \qquad \text{and} \qquad A^{-1^{\mathrm{T}}} = A^{\mathrm{T}^{-1}}$$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0.5 \end{bmatrix}$$

Top: $I, \quad A^{-1}, \quad A^{-1^{\mathrm{T}}}$

Bottom: $I, \quad A^{\mathrm{T}}, \quad A^{\mathrm{T}^{-1}}$

Undoing Maps: Inverse Matrices

How to compute A's inverse?
Start with

$$AA^{-1} = I$$

Denote two (unknown) columns of A^{-1} by $\bar{\mathbf{a}}_1$ and $\bar{\mathbf{a}}_2$
Denote columns of I by $\mathbf{e}_1$ and $\mathbf{e}_2$

$$A \begin{bmatrix} \bar{\mathbf{a}}_1 & \bar{\mathbf{a}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$$

Short for two linear systems

$$A\bar{\mathbf{a}}_1 = \mathbf{e}_1 \quad \text{and} \quad A\bar{\mathbf{a}}_2 = \mathbf{e}_2$$

Both systems have the same matrix A

Defining a Map

Matrices map vectors to vectors
If $\mathbf{v}_1$ and $\mathbf{v}_2$ mapped to $\mathbf{v}_1'$ and $\mathbf{v}_2'$, what matrix A did it?

$$A\mathbf{v}_1 = \mathbf{v}_1' \quad \text{and} \quad A\mathbf{v}_2 = \mathbf{v}_2'$$

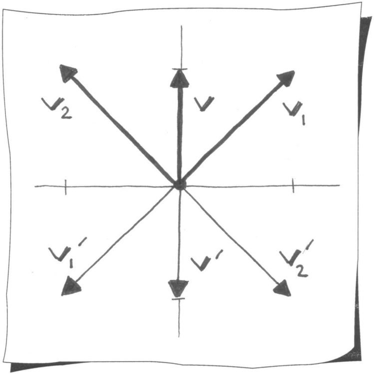
Combining:

$$A[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{v}_1', \mathbf{v}_2'] \qquad \text{or} \qquad AV = V'$$

Find V^{-1}, then $A = V'V^{-1}$

$\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent for V^{-1} to exist
If $\mathbf{v}_i$ and $\mathbf{v}_i'$ each linearly independent,
then A represents a change of basis

Change of basis example



$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}'_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}'_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Change of basis linear map

$$A = V'V^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

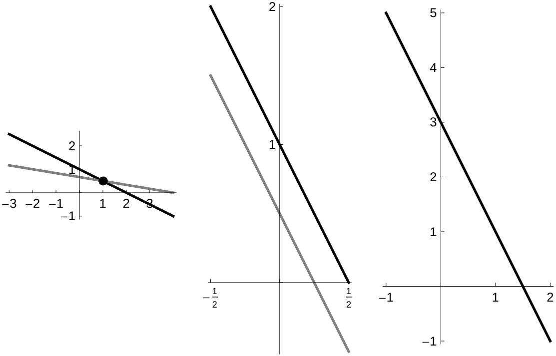
A Dual View

Two interpretations of of a linear system

1. "column view": coordinate system or linear combination approach
2. "row view": focus on the row equations

Line intersection problems provide examples of both:
— parametric/parametric line ⇒ column view
— implicit/implicit line ⇒ row view
Choose the view that best suits given information

A Dual View



Linear systems from this chapter interpreted as line intersection problems: unique solution, inconsistent, underdetermined

WYSK

- linear system
- solution spaces
- consistent linear system
- Cramer's rule
- upper triangular
- Gauss elimination
- forward elimination
- back substitution
- linear combination
- inverse matrix
- orthogonal matrix
- orthonormal
- rigid body motion
- inconsistent system of equations
- underdetermined system of equations
- homogeneous system
- kernel
- null space
- row pivoting
- column pivoting
- complete pivoting
- change of basis
- column and row views of linear systems