Practical Linear Algebra: A Geometry Toolbox

Fourth Edition

Chapter 7: Eigen Things

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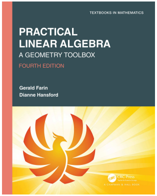


Outline

Tacoma Narrows Bridge:

Nov 1940 – swayed violently during mere 42-mile-per-hour winds
It collapsed seconds later

Linear map described by a matrix
Geometric properties?
— Phoenix figures showed circle mapped to ellipse: *action ellipse*

This stretching and rotating is the geometry of a linear map

Captured by its eigen things:
eigenvectors and *eigenvalues*

Introduction to Eigen Things

Tacoma Narrows Bridge: view from shore shortly before collapsing

Careful eigenvalue analysis carried-out before any bridge is built!

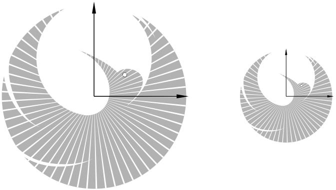
Eigenvalues and eigenvectors play important role in analysis of mechanical structures

Essentials of eigen-theory present in 2D case — topic of this chapter

Higher-dimensional case covered in Chapter 15

Fixed Directions

Uniform scaling: $\mathbf{e}_1$-axis is mapped to itself; $\mathbf{e}_2$-axis mapped to itself
$\Rightarrow$ Any vector $c\mathbf{e}_1$ or $d\mathbf{e}_2$ mapped to multiple of itself



Shear in $\mathbf{e}_1$: any vector $c\mathbf{e}_1$ mapped to multiple of itself

Fixed Directions

Fixed directions: directions not changed by the map

All vectors in fixed directions change only in length

Given matrix A: which vectors $\mathbf{r}$ mapped to a multiple of itself?

$$A\mathbf{r} = \lambda\mathbf{r} \qquad \lambda \in \mathbb{R}$$

Disregard the trivial solution $\mathbf{r} = \mathbf{0}$

In 2D: at most two directions
Symmetric matrices: directions orthogonal (more on that later)

Fixed directions called the eigenvectors
— from the German word "*eigen*" meaning special or proper
Factor λ called its eigenvalue

Eigen things are key to understanding geometry of a matrix

Eigenvalues

How to find the eigenvalues of a 2×2 matrix A?

$$A\mathbf{r} = \lambda\mathbf{r} = \lambda I\mathbf{r}$$

$$[A - \lambda I]\mathbf{r} = \mathbf{0}$$

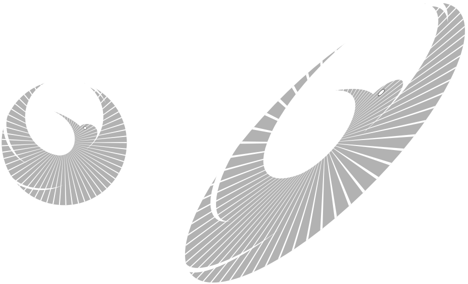
Matrix $[A - \lambda I]$ maps a nonzero vector $\mathbf{r}$ to the zero vector
$\Rightarrow [A - \lambda I]$ rank deficient matrix

$$\Rightarrow \quad p(\lambda) = \det[A - \lambda I] = 0$$

This is called the characteristic equation
— 2D: characteristic equation is quadratic

$p(\lambda)$ called the characteristic polynomial

Example:



$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$p(\lambda) = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

Recall quadratic equation:
$a\lambda^2 + b\lambda + c = 0$ has the solutions

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Eigenvalues

Eigenvalues of a 2×2 matrix:

Find the zeroes of the quadratic equation

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

Convention: eigenvalues ordered $|\lambda_1| \geq |\lambda_2|$

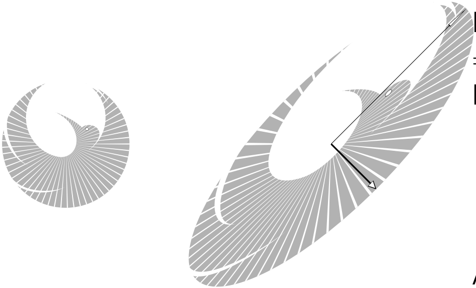
λ_1 called the dominant eigenvalue

$p(\lambda) = \det[A - \lambda I]$ then $p(0) = \det[A] = \lambda_1 \cdot \lambda_2$

Brings together concepts of the determinant and eigenvalues:
— Determinant measures change in area
(unit square mapped to parallelogram)
— Eigenvalues indicate a scaling of fixed directions

Eigenvectors

Example continued: Find $\mathbf{r}_1$ corresponding to $\lambda_1 = 3$



$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \mathbf{r}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{r}_1 = \mathbf{0}$$

Rank 1 homogeneous system
$\Rightarrow$ infinitely many solutions
Forward elimination results in

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{r}_1 = \mathbf{0}$$

Assign $r_{2,1} = 1$, then $\mathbf{r}_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
One-parameter family of eigenvectors forms the eigenspace for $\lambda_1 = 3$

Eigenvectors

Example continued: Find $\mathbf{r}_2$ corresponding to $\lambda_2 = 1$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \mathbf{r}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{r}_2 = \mathbf{0}$$

$$\mathbf{r}_2 = c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenspace corresponding to $\lambda_2 = 1$

Recheck Figure: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not stretched — it is mapped to itself

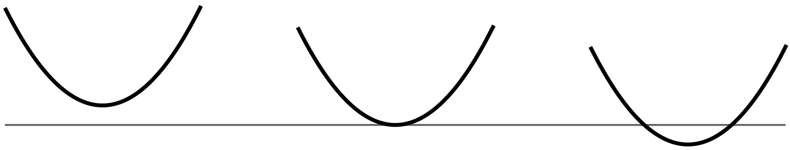
Often eigenvectors normalized for degree of uniqueness

$$\mathbf{r}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{r}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Dominant eigenvector: eigenvector corresponding to dominant eigenvalue

Striving for More Generality

Quadratic polynomials have either no, one, or two *real* zeroes



If there are no zeroes: then A has no fixed directions

Example: rotations — rotate every vector; no direction unchanged

Rotation by -90°

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0$$

No real solutions

Striving for More Generality

If there is one double root: then A has only one fixed direction

Example: A shear in the $\mathbf{e}_1$-direction

$$A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

Characteristic equation

$$\begin{vmatrix} 1-\lambda & 1/2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 1$$

To find the eigenvectors — solve

$$\begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} \mathbf{r} = \mathbf{0} \quad \Rightarrow \quad \text{(Column pivoting)} \quad \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} = \mathbf{0}$$

Set $r_1 = 1$, then $\mathbf{r} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Striving for More Generality

If one eigenvalue is zero: Example: projection matrix

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

Characteristic equation: $\lambda(\lambda - 1) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 0$
Eigenvector corresponding to λ_2:

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Forward elimination $\Rightarrow$ $\begin{bmatrix} 0.5 & 0.5 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{r}_2 = c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Matrix maps multiples of $\mathbf{r}_2$ to the zero vector
$\Rightarrow$ reduces dimensionality $\Rightarrow$ rank one

Eigenvector corresponding to zero eigenvalue is in the kernel or null space

Striving for More Generality

Projection matrix and eigenvalues:

Rank one matrices are idempotent: $A^2 = A$

One eigenvalue is zero – let $\lambda \neq 0$ with eigenvector $\mathbf{r}$

$$A\mathbf{r} = \lambda\mathbf{r}$$
$$A^2\mathbf{r} = \lambda A\mathbf{r}$$
$$\lambda\mathbf{r} = \lambda^2\mathbf{r}$$
$$\Rightarrow \lambda = 1$$

A 2D projection matrix always has eigenvalues 0 and 1

General statement: a 2×2 matrix with one zero eigenvalue is rank 1

The Geometry of Symmetric Matrices

Symmetric matrices: $A = A^{\mathrm{T}}$

Arise often in practical problems
— examples: conics and least squares approximation
Many more practical examples in classical mechanics, elasticity theory, quantum mechanics, and thermodynamics

Real symmetric matrices advantages:

- eigenvalues are real
- interesting geometric interpretation (eigendecomposition — next)
- structure allows for stable and efficient numerical algorithms

The Geometry of Symmetric Matrices

Two basic equations for eigenvalues and eigenvectors:

$$A\mathbf{r}_1 = \lambda_1 \mathbf{r}_1 \quad (*) \qquad\qquad A\mathbf{r}_2 = \lambda_2 \mathbf{r}_2 \quad (**)$$

Since A is symmetric

$$(A\mathbf{r}_1)^{\mathrm{T}} = (\lambda_1 \mathbf{r}_1)^{\mathrm{T}}$$
$$\mathbf{r}_1^{\mathrm{T}} A^{\mathrm{T}} = \mathbf{r}_1^{\mathrm{T}} \lambda_1$$
$$\mathbf{r}_1^{\mathrm{T}} A = \lambda_1 \mathbf{r}_1^{\mathrm{T}}$$

Multiply both sides by $\mathbf{r}_2$

$$\mathbf{r}_1^{\mathrm{T}} A \mathbf{r}_2 = \lambda_1 \mathbf{r}_1^{\mathrm{T}} \mathbf{r}_2$$

Multiply both sides of (**) by $\mathbf{r}_1^{\mathrm{T}}$

$$\mathbf{r}_1^{\mathrm{T}} A \mathbf{r}_2 = \lambda_2 \mathbf{r}_1^{\mathrm{T}} \mathbf{r}_2$$

Equating last two equations

$$\lambda_1 \mathbf{r}_1^{\mathrm{T}} \mathbf{r}_2 = \lambda_2 \mathbf{r}_1^{\mathrm{T}} \mathbf{r}_2 \qquad \text{or} \qquad (\lambda_1 - \lambda_2)\mathbf{r}_1^{\mathrm{T}} \mathbf{r}_2 = 0$$

The Geometry of Symmetric Matrices

$$(\lambda_1 - \lambda_2)\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = 0$$

If $\lambda_1 \neq \lambda_2$ (the standard case): $\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = 0 \Rightarrow$ *orthogonal*
Condense $(*)$ and $(**)$ into one matrix equation

$$\begin{bmatrix} A\mathbf{r}_1 & A\mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{r}_1 & \lambda_2\mathbf{r}_2 \end{bmatrix}$$

Define

$$R = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

then

$$AR = R\Lambda$$

Revisit Example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

The Geometry of Symmetric Matrices

Assume eigenvectors are normalized: $\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_1 = 1$ and $\mathbf{r}_2^{\mathrm{T}}\mathbf{r}_2 = 1$
They are orthogonal: $\mathbf{r}_1^{\mathrm{T}}\mathbf{r}_2 = \mathbf{r}_2^{\mathrm{T}}\mathbf{r}_1 = 0$
Two conditions $\Rightarrow$ $\mathbf{r}_1$ and $\mathbf{r}_2$ are *orthonormal*

These four equations written in matrix form

$$R^{\mathrm{T}}R = I \quad \Rightarrow \quad R^{-1} = R^{\mathrm{T}} \qquad R \text{ is an orthogonal matrix}$$

Now $AR = R\Lambda$ becomes

$$A = R\Lambda R^{\mathrm{T}}$$

The *eigendecomposition* of A
May transform A to diagonal matrix $\Lambda = R^{-1}AR$: A is diagonalizable
Matrix decomposition: fundamental tool in linear algebra
— gives insight into the action of a matrix
— for building stable and efficient methods to solve linear systems

The Geometry of Symmetric Matrices

Geometric meaning of the eigendecomposition $A = R\Lambda R^{\mathrm{T}}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{with} \quad \lambda_1 = 3, \quad \lambda_2 = 1$$

Top: I, A

Bottom: I, R^{T} (rotate $-45°$), ΛR^{T} (scale), $R\Lambda R^{\mathrm{T}}$ (rotate $45°$)

R: rotation, a reflection, or combination $\Rightarrow$ R^{T}: reversal of R

These linear maps preserve lengths and angles

Diagonal matrix Λ is a scaling along each of the coordinate axes

The Geometry of Symmetric Matrices

Another look at the action of the map A on a vector $\mathbf{x}$:

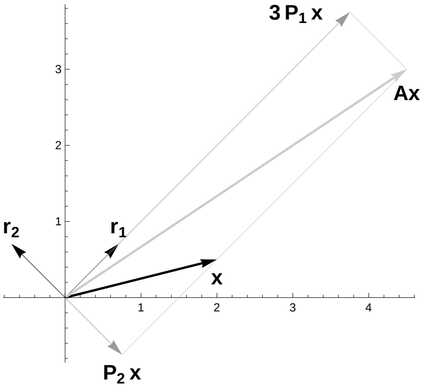
$$\begin{aligned} A\mathbf{x} &= R\Lambda R^{\mathrm{T}}\mathbf{x} \\ &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix} \Lambda \begin{bmatrix} \mathbf{r}_1^{\mathrm{T}} \\ \mathbf{r}_2^{\mathrm{T}} \end{bmatrix} \mathbf{x} \\ &= \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \mathbf{r}_1^{\mathrm{T}}\mathbf{x} \\ \lambda_2 \mathbf{r}_2^{\mathrm{T}}\mathbf{x} \end{bmatrix} \\ &= \lambda_1 \mathbf{r}_1 \mathbf{r}_1^{\mathrm{T}}\mathbf{x} + \lambda_2 \mathbf{r}_2 \mathbf{r}_2^{\mathrm{T}}\mathbf{x} \end{aligned}$$

Each matrix $\mathbf{r}_k\mathbf{r}_k^{\mathrm{T}}$ is a projection onto $\mathbf{r}_k$

Action of A can be interpreted as a linear combination of projections onto the orthogonal eigenvectors

Example: action of A on $\mathbf{x}$ as a linear combination of projections

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$



Projection matrices:

$$P_1 = \mathbf{r}_1\mathbf{r}_1^{\mathrm{T}} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P_2 = \mathbf{r}_2\mathbf{r}_2^{\mathrm{T}} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Action of the map:

$$A\mathbf{x} = 3P_1\mathbf{x} + P_2\mathbf{x}$$

$$= \begin{bmatrix} 15/4 \\ 15/4 \end{bmatrix} + \begin{bmatrix} 3/4 \\ -3/4 \end{bmatrix} = \begin{bmatrix} 9/2 \\ 3 \end{bmatrix}$$

Quadratic Forms

Bivariate function: a function f with two arguments $f(v_1, v_2)$ or $f(\mathbf{v})$
Special bivariate functions defined in terms of a 2×2 symmetric matrix C:

$$f(\mathbf{v}) = \mathbf{v}^{\mathrm{T}} C \mathbf{v}$$

Such functions are called quadratic forms — all terms are quadratic:

$$f(\mathbf{v}) = c_{1,1} v_1^2 + 2c_{2,1} v_1 v_2 + c_{2,2} v_2^2$$

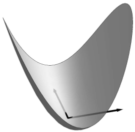
Graph of a quadratic form is a 3D point set $[v_1, v_2, f(v_1, v_2)]^{\mathrm{T}}$
forming a quadratic surface

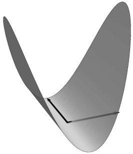
Quadratic Forms

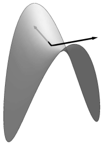
Ellipsoid, paraboloid, hyperboloid evaluated over the unit circle

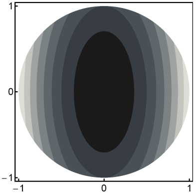
Contour plot communicates additional shape information

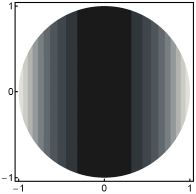
Color map extents: min $f(\mathbf{v})$ colored black and max $f(\mathbf{v})$ colored white

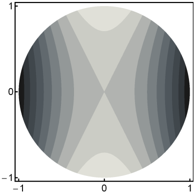




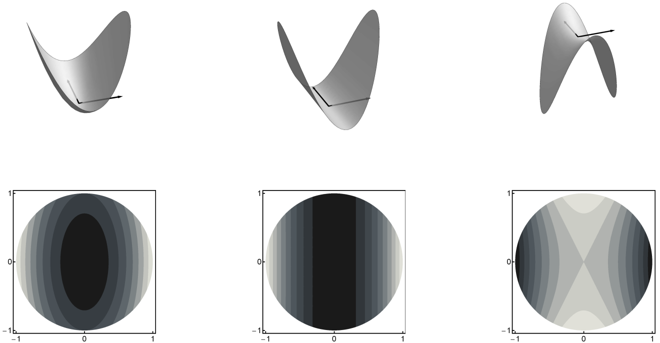








Quadratic Forms



Corresponding matrices and quadratic forms are

$$C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad C_3 = \begin{bmatrix} -2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$f_1(\mathbf{v}) = 2v_1^2 + 0.5v_2^2 \qquad f_2(\mathbf{v}) = 2v_1^2 \qquad f_3(\mathbf{v}) = -2v_1^2 + 0.5v_2^2$$

Quadratic Forms

$$C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \quad C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad C_3 = \begin{bmatrix} -2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Determinant and eigenvalues:

$$\begin{aligned} |C_1| &= 1 & \lambda_1 &= 2, & \lambda_2 &= 0.5 \\ |C_2| &= 0 & \lambda_1 &= 2, & \lambda_2 &= 0 \\ |C_3| &= -1 & \lambda_1 &= -2, & \lambda_2 &= 0.5 \end{aligned}$$

Quadratic Forms

Positive definite matrix:

$$f(\mathbf{v}) = \mathbf{v}^{\mathrm{T}} A \mathbf{v} > 0 \quad \text{for } \mathbf{v} \neq \mathbf{0} \in \mathbb{R}^2$$

Quadratic form is positive everywhere except for $\mathbf{v} = \mathbf{0}$

Example: ellipsoid $\quad C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad f_1(\mathbf{v}) = 2v_1^2 + 0.5v_2^2$

Positive definite symmetric matrices: special class of matrices

— arise in a number of applications

— lend themselves to numerically stable and efficient algorithms

Quadratic Forms

Suppose A is not necessarily symmetric

$A^{\mathrm{T}}A$ and AA^{T} are symmetric and positive definite

$$\mathbf{v}^{\mathrm{T}}A^{\mathrm{T}}A\mathbf{v} = (A\mathbf{v})^{\mathrm{T}}(A\mathbf{v}) = \mathbf{y}^{\mathrm{T}}\mathbf{y} = ||\mathbf{y}||^2 > 0$$

These matrices at the heart the singular value decomposition (SVD)
— topic of Chapter 16

Determinant of a positive definite 2×2 matrix is always positive
— This matrix is always nonsingular

These concepts apply to $n \times n$ matrices, however there are additional requirements on the determinant
— More detail in Chapters 12 and 15

Quadratic Forms

Examine quadratic forms where C is positive definite: $C = A^{\mathrm{T}}A$

Contour: all $\mathbf{v}$ for which

$$\mathbf{v}^{\mathrm{T}}C\mathbf{v} = 1$$

Example: contour for C_1 is an ellipse $2v_1^2 + 0.5v_2^2 = 1$

Set $v_1 = 0 \quad \Rightarrow$ $\mathbf{e}_2$-axis extents of the ellipse: $\pm 1/\sqrt{0.5}$

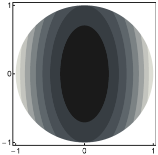
Set $v_2 = 0 \quad \Rightarrow$ $\mathbf{e}_1$-axis extents: $\pm 1/\sqrt{2}$

Major axis is the longest — here: $\mathbf{e}_2$-direction

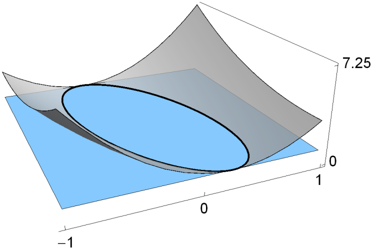
Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 0.5$

Eigenvectors $\mathbf{r}_1 = [1 \;\; 0]^{\mathrm{T}}$, $\mathbf{r}_2 = [0 \;\; 1]^{\mathrm{T}}$

$\Rightarrow$ Minor axis corresponds to the dominant eigenvector – steeper ascent



Example:



$$A = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix} \qquad C_4 = A^{\mathrm{T}} A = \begin{bmatrix} 4 & 1 \\ 1 & 1.25 \end{bmatrix}$$

Eigendecomposition $C_4 = R \Lambda R^{\mathrm{T}}$

$$R = \begin{bmatrix} -0.95 & 0.30 \\ -0.30 & -0.95 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 4.3 & 0 \\ 0 & 0.92 \end{bmatrix}$$

Ellipse defined by $\mathbf{v}^{\mathrm{T}} C_4 \mathbf{v} = 4v_1^2 + 2v_1 v_2 + 1.25v_2^2 = 1$

Quadratic Forms

Example continued:

$$\text{Ellipse} \quad \mathbf{v}^{\mathrm{T}} C_4 \mathbf{v} = 4v_1^2 + 2v_1 v_2 + 1.25 v_2^2 = 1$$

Major and minor axis not aligned with coordinate axes

Fnd major and minor axis lengths:
— use eigendecomposition to perform a coordinate transformation
— align ellipse with the coordinate axes

$$\mathbf{v}^{\mathrm{T}} R \Lambda R^{\mathrm{T}} \mathbf{v} = 1$$
$$\hat{\mathbf{v}}^{\mathrm{T}} \Lambda \hat{\mathbf{v}} = 1$$
$$\lambda_1 \hat{v}_1^2 + \lambda_2 \hat{v}_2^2 = 1$$

Minor axis: length $1/\sqrt{\lambda_1} = 0.48$ on $\mathbf{e}_1$ axis
Major axis: length $1/\sqrt{\lambda_2} = 1.04$ on $\mathbf{e}_2$ axis

Quadratic Forms

Quadratic form for a symmetric matrix may also be written as

$$q(\mathbf{v}) = \frac{\mathbf{v}^{\mathrm{T}} C \mathbf{v}}{\mathbf{v}^{\mathrm{T}} \mathbf{v}}$$

Called the Rayleigh quotient

Quotient reaches a maximum in the dominant eigenvector direction

$$q(\mathbf{r}_1) = \frac{\mathbf{r}_1^{\mathrm{T}} C \mathbf{r}_1}{\mathbf{r}_1^{\mathrm{T}} \mathbf{r}_1} = \frac{\mathbf{r}_1^{\mathrm{T}} \lambda_1 \mathbf{r}_1}{\mathbf{r}_1^{\mathrm{T}} \mathbf{r}_1} = \lambda_1$$

and a minimum in the direction corresponding to the smallest eigenvalue

$$q(\mathbf{r}_2) = \frac{\mathbf{r}_2^{\mathrm{T}} C \mathbf{r}_2}{\mathbf{r}_2^{\mathrm{T}} \mathbf{r}_2} = \frac{\mathbf{r}_2^{\mathrm{T}} \lambda_2 \mathbf{r}_2}{\mathbf{r}_2^{\mathrm{T}} \mathbf{r}_2} = \lambda_2$$

Algebraic confirmation of the steepest ascent observation

Repeating Maps

Matrices map the unit circle to an ellipse

Map the ellipse using the same map — Repeat

$$A = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}$$

Repeating Maps

$$A = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}$$

Symmetric $\Rightarrow$ two real eigenvalues and orthogonal eigenvectors

As map repeated resulting ellipses stretched:

— elongated in direction $\mathbf{r}_1$ by $\lambda_1 = 1.3$

— compacted in the direction of $\mathbf{r}_2$ by a factor of $\lambda_2 = 0.7$

$$\mathbf{r}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$AA\mathbf{r}_1 = A\lambda_1\mathbf{r}_1 = \lambda_1^2\mathbf{r}_1$$

$$A^n\mathbf{r}_1 = \lambda_1^n\mathbf{r}_1$$

Same holds for $\mathbf{r}_2$ and λ_2

Repeating Maps

$$A = \begin{bmatrix} 0.7 & 0.3 \\ -1 & 1 \end{bmatrix}$$

Matrix does not have real eigenvalues — related to a rotation matrix — figures do not line up along any (real) fixed directions

Power method: apply idea of repeating a map to find the dominant eigenvector (Chapter 15)

WYSK

- fixed direction
- eigenvalue
- eigenvector
- characteristic equation
- dominant eigenvalue
- dominant eigenvector
- homogeneous system
- kernel or null space
- orthogonal matrix
- eigen-theory of a symmetric matrix
- matrix with real eigenvalues
- diagonalizable matrix
- eigendecomposition
- quadratic form
- contour plot
- positive definite matrix
- repeated linear map