Eigenvalues and Functions

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An eigenvalue λ of a matrix A is typically thought of as a solution of the matrix equation

$$A\mathbf{x} = \lambda\mathbf{x}.$$

In Section 15.6, we encountered more general spaces than those formed by finite-dimensional vectors: those are spaces formed by *polynomials*. Now, we will even go beyond that: we will explore the space of all real-valued functions. Do eigenvalues and eigenvectors have meaning there? Let's see.

Let f be a function, meaning that $y = f(x)$ assigns the output value y to an input value x, and we assume both x and y are real numbers. We also assume that f is smooth, or differentiable.

The set of all such functions f forms a linear space. This is easily checked by verifying that the zero function is in the space and that linearity is obeyed: if f and g are in the space, then so is $\alpha f + \beta g$ with α and β being real numbers. This space is different from the ones we have encountered so far: its elements are functions, not vectors. But that does not really matter, what is important is that is has the *structure* of a linear space.

We can define linear maps L for elements of this function space. For example, setting $Lf = 2f$ is such a map, albeit a bit trivial. A more interesting linear map (see Section 15.6) is that of taking derivatives: $Df = f'$.

How can we marry the concept of eigenvalues and linear maps such as D? First of all, D will not have eigen*vectors*, since our linear space consists of functions, not vectors. So we will talk of eigen*functions* instead. A function f is an eigenfunction of D if it is mapped to a multiple of itself:

$$Df = \lambda f.$$

The scalar multiple λ is, as usual, referred to as f's eigenvalue. Note that D may have many eigenfunctions, each corresponding to a different λ.

Since we know that D means taking derivatives, this becomes

$$f' = \lambda f. \tag{1}$$

Any function f satisfying (1) is thus an eigenfunction of the derivative map D.

Now you have to recall your calculus: the function $f(x) = e^{\lambda x}$ satisfies (for $\lambda \neq 0$):

$$f'(x) = \lambda e^{\lambda x}$$

which may be written as

$$Df = \lambda f.$$

Hence all real numbers $\lambda \neq 0$ are eigenvalues of D, and the corresponding eigenfunctions are $e^{\lambda x}$. All multiples $ce^{\lambda x}$ are eigenfunctions as well. We see that our map D has infinitely many eigenfunctions!

We may generalize this concept: any linear map of the form $Lf = \sum \alpha_i f^{(i)}$ will have eigenfunctions, but we will not explore that here.

This may seem a bit abstract but actually has many uses for example in differential equations.