A Practical Approach to Teaching Linear Algebra

BY GERALD FARIN AND DIANNE HANSFORD

Both of us teach Computer Graphics and Geometric Modeling at the senior year college level. The most important prerequisite: Linear Algebra. A big problem: students have a dim recollection of Linear Algebra at best. Dot products and matrices are concepts that need to be re-taught as we go.

Why is this? We believe it is because the standard way of teaching Linear Algebra is not suited for retaining the material. Example: the action of a matrix is easily explained in algebraic terms. But, to motivate it, a simple 2D image like the one in Figure 1 would do a better job. For most students, illustrations and geometry make for better retention than equations and formalisms. Thus was born the concept for a more geometric and applications oriented approach, leading to the book *Practical Linear Algebra — A Geometry Toolbox, PLA* for short.

A $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$

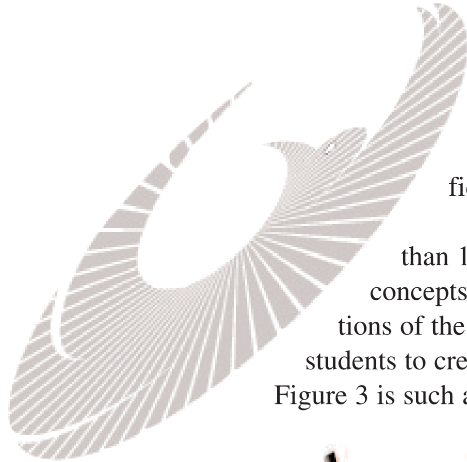


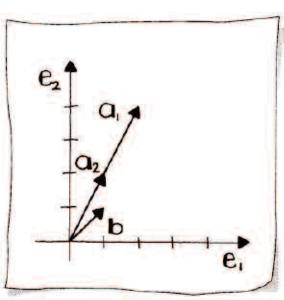
Figure 1

Left: a 2D object. Right: the result of applying the given matrix to the object.

Computer graphics is of interest to most teens and twenty-somethings, whether it is in the form of gaming or special effects in movies and commercials. Additionally, the basic concepts are accessible via an introductory level of Linear Algebra; Calculus isn't necessary. For these reasons, we chose computer graphics as the primary application to motivate the material in *PLA*. Giving students an inside look at how games and special effects work is sure to captivate their interest. An accessible and interesting motivator gives students an opportunity to discuss key ideas in terms that interest them. For instance, they might discuss the flips and turns made by their avatar in a game, but now they have the vocabulary to speak in terms of matrices. From our experience, materials presented intuitively and graphically are more easily comprehended and will be far better retained.

PLA takes a very visual approach. More than 100 hand-drawn sketches fill the margins. An example is shown in Figure 2. And students are encouraged to make sketches of their own. Sketching is a method of self-teaching, and it further develops one's communication skills. Additionally, this proactive, visual approach gives students a means to verify correctness of algebraic solutions, as well as gain confidence in their own work.

In addition to the sketches, more than 100 figures illustrate more advanced concepts; sometimes, they are just fun applications of the concepts. Hopefully the figures inspire students to create their own geometric creations. Figure 3 is such an example.



$$\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

Figure 2

Left: a 2D object. Right: the result of applying the given matrix to the object.

Figure 3

A sample figure from PLA which is used to illustrate 3D affine maps in action.

PLA is more than just a Linear Algebra book – it is a Geometry Toolbox. It gives students the tools necessary for further studies in engineering and technology. Topics that marry nicely with Linear Algebra but give broader geometric knowledge include parametric and implicit representations of lines and planes, intersections of lines and planes, working with triangles and polygons, conic sections, and parametric curves.

While being practical and visual, *PLA* also does not miss out on the "bread-and-butter" topics. Subspaces, function spaces, eigenvectors, linear systems – they are all covered, and wherever possible, with illustrations and some intuition.

Linear Algebra is at the core of nearly all engineering and technology domains. It is vital that we produce students that can master Linear Algebra, but to do this we must first grab their attention. Computer graphics touches our lives every day in the form of gaming, special effects in movies and commercials, user interfaces (windowing systems), and scientific visualization (weather on the news). It has become truly ubiquitous. And as our world becomes more visual, its influence will continue to grow. *PLA* has employed computer graphics as the motivator, and we believe this will lead to better retention of the mathematics necessary to be successful in engineering and technology. If more programs adopt teaching methodologies as we are promoting in *PLA*, we believe that in the future computer graphics instructors won't have to devote one-sixth of a class to rudimentary Linear Algebra material.

PLA is also practical for teachers: upon request we supply solutions to all Problems in the text, sample exams (with solutions), and we provide more examples. Additionally, all the sketches and figures are available on the book's website.

LINEAR ALGEBRA IN HIGH SCHOOL PROGRAMS

Linear Algebra in the style of *PLA* offers an introduction to 2D and 3D geometry that will give students the foundations that are so very helpful for freshman science and engineering courses. Unfortunately, the science and mathematics curriculum in Universities is out of sync. Students are presented with applications, for example mechanics in physics, without the necessary geometry toolkit. They are forced to learn geometry and physics concurrently, and this can be overwhelming.

From our experience, *PLA*'s approach to learning, which combines algebraic theory with geometry, particularly in the form of figures and sketches, will prove to be a valuable tool in a number of University courses.

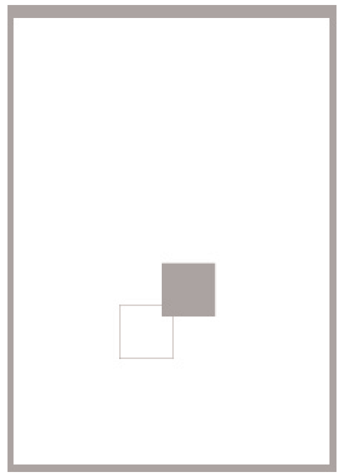
We hope that this heavily illustrated text captures the imagination of students, and excites them enough to pursue science and technology at University. Also, by connecting the theory of Linear Algebra with its application in computer graphics, we hope students see the relevance of the material early in their education.

Gerald Farin and Dianne Hansford teach Computer Science and Engineering at Arizona State University Tempe, AZ 85287-8806 dianne.hansford}@asu.edu and farin@asu.edu

Their book, Practical Linear Algebra A Geometry Toolbox, is available through A K Peters http://akpeters.com. It is 384 pages, and was published in 2005.

PROJECT IDEAS

This visual approach of *PLA* can be expanded by getting the students involved with an easy-to-use drawing language called PostScript. All the figures were generated with Postscript, and they are available on the book's website. A tutorial is provided in the Appendix, and using PostScript is free. This is a great way to begin with Linear Algebra via basic computer graphics concepts. By exploring the examples provided in *PLA*, and by modifying given PostScript "programs", students can become familiar with a simple programming language and create neat images of their own. This gives them a fun and exciting method to explore Linear Algebra concepts disguised in computer graphics. Figure 4 illustrates how simple PostScript is to use.



```
%!
% A sample PostScript program.
%

% draw a box
newpath
        200 200 moveto
        300 200 lineto
        300 300 lineto
        200 300 lineto
        200 200 lineto
stroke

% translate the box and color it in

80 80 translate
0.5 setgray

newpath
        200 200 moveto
        300 200 lineto
        300 300 lineto
        200 300 lineto
        200 200 lineto
fill

showpage
```

Figure 4

Right: a sample PostScript program. Left: the display of the program. The lower-left corner of the page is the origin.

SIDE/INSET BOX MATERIAL

PLA makes an effort to bring a visual approach to each topic; even to eigenvalues and eigenvectors! Figure 5 is a sample illustration from the chapter on "Eigenthings", demonstrating that real eigenvectors play a more and more prominent role as a matrix is applied repeatedly.

$$A = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}$$

Figure 5

Repeatedly applying the matrix A results in the eigenvectors and eigenvalues to become more prominent. The eigenvectors are [1,1]T and [-1,1]T and the corresponding eigenvalues are 1.3 and 0.7, respectively.