Practical Linear Algebra: A Geometry Toolbox
Third edition

Chapter 14: General Linear Spaces

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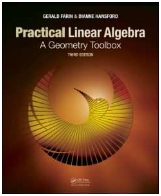


Outline

General Linear Spaces

All cubic polynomials over the interval [0,1] form a linear space
Some elements illustrated

Linear space = vector space
Chapters 4 and 9: examined properties in 2D and 3D

Here: higher dimensions
— Spaces can be abstract
— Powerful concept in dealing with real-life problems

- car crash simulations
- weather forecasts
- computer games

"General" refers to the dimension and abstraction

Basic Properties of Linear Spaces

$\mathcal{L}_n$: linear space of dimension n

Elements of $\mathcal{L}_n$ are vectors
— Denoted by boldface letters such as $\mathbf{u}$

Two operations defined on the elements of $\mathcal{L}_n$:
— Addition
— Multiplication by a scalar

Linearity property
Any *linear combination* of vectors results in a vector in the same space

$$\mathbf{w} = s\mathbf{u} + t\mathbf{v}$$

Both s and t may be zero $\Rightarrow$ every linear space has a zero vector in it

Basic Properties of Linear Spaces

Generalize linear spaces: include new kinds of vectors
— Objects in the linear space are not always in traditional vector format
— Key: the linearity property

Example: $\mathbb{R}^2$

$$\text{Elements of space:} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{w} = 2\mathbf{u} + \mathbf{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \text{is also in } \mathbb{R}^2$$

Example: Linear space $\mathcal{M}_{2\times 2}$ – the set of all 2×2 matrices
— Rules of matrix arithmetic guarantee the linearity property

Example: $\mathcal{V}_2$ – all vectors $\mathbf{w}$ in $\mathbb{R}^2$ that satisfy $w_2 \geq 0$
— $\mathbf{e}_1$ and $\mathbf{e}_2$ live in $\mathcal{V}_2$ — Is this a linear space?

$$\text{No: } \mathbf{v} = 0 \times \mathbf{e}_1 + -1 \times \mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{which is not in} \mathcal{V}_2$$

Basic Properties of Linear Spaces

In $\mathcal{L}_n$ define a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ where $1 \leq r \leq n$

Vectors are linearly independent means

$$\mathbf{v}_1 = s_2\mathbf{v}_2 + s_3\mathbf{v}_3 + \ldots + s_r\mathbf{v}_r$$

Will *not* have a solution set $s_2, \ldots, s_r$

$\Rightarrow$ Zero vector can only be expressed in a trivial manner:

$$\text{If} \quad \mathbf{0} = s_1\mathbf{v}_1 + \ldots + s_r\mathbf{v}_r \quad \text{then } s_1 = \ldots = s_r = 0$$

If the zero vector *can* be expressed as a nontrivial combination of r vectors then these vectors are linearly dependent

Basic Properties of Linear Spaces

Subspace of $\mathcal{L}_n$ of dimension r:
Formed from all *linear combinations* of linearly independent $\mathbf{v}_1, \ldots, \mathbf{v}_r$
$\Rightarrow$ Subspace is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_r$

If this subspace equals whole space $\mathcal{L}_n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a basis for $\mathcal{L}_n$

If $\mathcal{L}_n$ is a linear space of dimension n
then any $n+1$ vectors in it are linearly dependent

Example: $\mathbb{R}^3$ and basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{is also in } \mathbb{R}^3$$

The four vectors $\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly dependent

Any one of four vectors forms a *one-dimensional subspace* of $\mathbb{R}^3$
Any two vectors here form a *two-dimensional subspace* of $\mathbb{R}^3$

Basic Properties of Linear Spaces

Example: $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

These vectors are linearly dependent since

$$\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_3 \quad \text{or} \quad \mathbf{0} = \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$$

Set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains only two linearly independent vectors
$\Rightarrow$ Any two of them spans a subspace of $\mathbb{R}^4$ of dimension two

Example: $\mathbb{R}^3$

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

These four vectors are linearly dependent since

$$\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4$$

Any set of three of these vectors is a basis for $\mathbb{R}^3$

Linear Maps

$A : \mathcal{L}_n \to \mathcal{L}_m$ — The linear map A that transforms $\mathcal{L}_n$ to $\mathcal{L}_m$

A preserves linear relationships
Preimage $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathcal{L}_n$ mapped to image $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$ in $\mathcal{L}_m$

$$\mathbf{v}_1 = \alpha \mathbf{v}_2 + \beta \mathbf{v}_3 \quad \Rightarrow \quad A\mathbf{v}_1 = \alpha A\mathbf{v}_2 + \beta A\mathbf{v}_3$$

Maps without this property: nonlinear maps

Linear map: $m \times n$ matrix A
$\mathbf{v}$ in $\mathcal{L}_n \to \mathbf{v}'$ in $\mathcal{L}_m \quad \Rightarrow \mathbf{v}' = A\mathbf{v}$
$A : [\mathbf{e}_1, \ldots, \mathbf{e}_n]$-system $\to [\mathbf{a}_1, \ldots, \mathbf{a}_n]$-system
$\Rightarrow$

$$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \ldots v_n\mathbf{a}_n \quad \text{is in the column space of } A$$

Linear Maps

Example: $A : \mathbb{R}^2 \to \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$$

Given vectors in $\mathbb{R}^2$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

mapped to vectors in $\mathbb{R}^3$

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad \hat{\mathbf{v}}_3 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

$\mathbf{v}_i$ are *linearly dependent* since $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$
Linear maps preserve linear relationships $\Rightarrow \mathbf{v}'_3 = 2\mathbf{v}'_1 + \mathbf{v}'_2$

Matrix rank

$m \times n$ matrix can be at most of rank $k = \min\{m, n\}$
Rank equals number of linearly independent column vectors
If $\text{rank}(A) = \min\{m, n\} \Rightarrow$ full rank
If $\text{rank}(A) < \min\{m, n\} \Rightarrow$ rank deficient

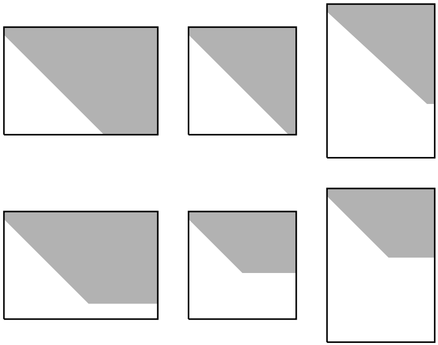
Linear map can never *increase* dimension
— Possible to map $\mathcal{L}_n$ to higher-dimensional space $\mathcal{L}_m$
Images of $\mathcal{L}_n$'s n basis vectors will span
a subspace of $\mathcal{L}_m$ of dimension at most n
(See last Example)

How to identify rank?
Perform forward elimination until matrix in upper triangular form
— k nonzero rows $\Rightarrow$ rank is k

Linear Maps

Rank scenarios for an $m \times n$ matrix
Matrices in upper triangular form

$m < n$ $\quad$ $m = n$ $\quad$ $m > n$



Top row: full rank matrices
Bottom row: rank deficient matrices

Example: Determine the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{Forward elimination} \quad \Rightarrow \quad \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

One row of zeroes: matrix has rank 3 — full rank since $min\{4, 3\} = 3$
Example: Determine the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{Forward elimination} \quad \Rightarrow \quad \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix has rank 2 — rank deficient

Review features of linear maps from earlier chapters

$n \times n$ matrix A that is rank n is *invertible*
$\Rightarrow$ *inverse matrix* A^{-1} exists

If A is invertible then it does not reduce dimension
$\Rightarrow$ *Determinant* is nonzero
— Measures volume of nD parallelepiped defined by columns vectors
— Computed by transforming matrix to upper triangular
(via shears/forward elimination)
Then the determinant is the product of the diagonal elements
(pivoting: careful of sign)

Inner Products

Inner product: a map from $\mathcal{L}_n$ to the reals $\mathbb{R}$ — denoted as $\langle \mathbf{v}, \mathbf{w} \rangle$

Properties:

Symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$

Homogeneity: $\langle \alpha\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{w}, \mathbf{v} \rangle$

Additivity: $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}$ $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$

Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

Homogeneity and additivity properties combined:

$$\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha\langle \mathbf{u}, \mathbf{w} \rangle + \beta\langle \mathbf{v}, \mathbf{w} \rangle$$

Example: the *dot product* $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

Inner product space: a linear space with an inner product

Inner Products

Example: Define a "test" inner product in $\mathbb{R}^2$

$$\langle \mathbf{v}, \mathbf{w} \rangle = 4v_1 w_1 + 2v_2 w_2$$

Compare it to the dot product:

$$\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = 4(1)(0) + 2(0)(1) = 0 \qquad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0$$

Let $\mathbf{r} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (unit vector)

$$\langle \mathbf{e}_1, \mathbf{r} \rangle = 4(1)(\frac{1}{\sqrt{2}}) + 2(0)(\frac{1}{\sqrt{2}}) = \frac{4}{\sqrt{2}} \qquad \mathbf{e}_1 \cdot \mathbf{r} = \frac{1}{\sqrt{2}}$$

Inner Products

Does the test inner product satisfy the necessary properties?

Symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = 4v_1 w_1 + 2v_2 w_2 = 4w_1 v_1 + 2w_2 v_2 = \langle \mathbf{w}, \mathbf{v} \rangle$

Homogeneity: $\langle \alpha\mathbf{v}, \mathbf{w} \rangle = 4(\alpha v_1)w_1 + 2(\alpha v_2)w_2 = \alpha(4v_1 w_1 + 2v_2 w_2) = \alpha\langle \mathbf{v}, \mathbf{w} \rangle$

$$\begin{aligned}\text{Additivity: } \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 4(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= (4u_1 w_1 + 2u_2 w_2) + (4v_1 w_1 + 2v_2 w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle\end{aligned}$$

Positivity: $\langle \mathbf{v}, \mathbf{v} \rangle = 4v_1^2 + 2v_2^2 \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$

Usefulness of this inner product? But it does satisfy the properties!

Length

2-norm or Euclidean norm: $\|\mathbf{v}\|_2 = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$
("Usual" norm $\Rightarrow$ subscript typically omitted)

Distance between two vectors

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle} = \|\mathbf{u} - \mathbf{v}\|$$

Example: the dot product in $\mathbb{R}^n$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

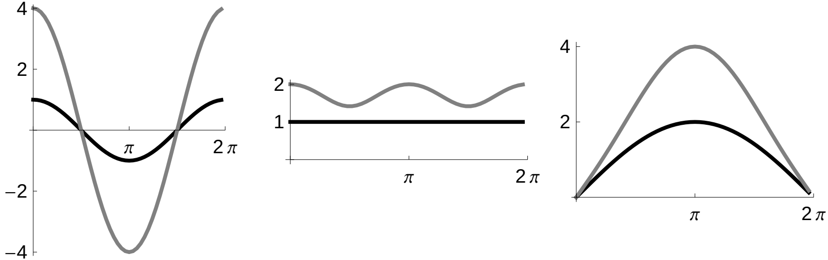
$$\text{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

Inner Products

Norm and distance for two inner products

Test inner product	**Dot product**
$\langle \mathbf{v}, \mathbf{w} \rangle = 4v_1w_1 + 2v_2w_2$	$\langle \mathbf{v}, \mathbf{w} \rangle = v_1w_1 + v_2w_2$
$\lVert \mathbf{e}_1 \rVert = \sqrt{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle} = 4(1)^2 + 2(0)^2 = 4$	$\lVert \mathbf{e}_1 \rVert = 1$
$\mathrm{dist}(\mathbf{e}_1, \mathbf{e}_2) = \sqrt{4(1-0)^2 + 2(0-1)^2} = \sqrt{6}$	$\mathrm{dist}(\mathbf{e}_1, \mathbf{e}_2) = \sqrt{2}$

Black: dot product Gray: test inner product $\langle \mathbf{v}, \mathbf{w} \rangle = 4v_1 w_1 + 2v_2 w_2$



Unit vector $\mathbf{r}$ rotated $[0, 2\pi]$

Left: inner product $\mathbf{e}_1 \cdot \mathbf{r}$ and $\langle \mathbf{e}_1, \mathbf{r} \rangle$

Middle: length $\sqrt{\mathbf{r} \cdot \mathbf{r}}$ and $\sqrt{\langle \mathbf{r}, \mathbf{r} \rangle}$

Right: distance $\sqrt{(\mathbf{e}_1 - \mathbf{r}) \cdot (\mathbf{e}_1 - \mathbf{r})}$ and $\sqrt{\langle (\mathbf{e}_1 - \mathbf{r}), (\mathbf{e}_1 - \mathbf{r}) \rangle}$

Inner Products

Orthogonality: $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for $\mathbf{v}, \mathbf{w}$ in $\mathcal{L}_n$

Orthogonal basis: $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for $\mathcal{L}_n$
and all $\mathbf{v}_i$ are mutually orthogonal: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$

And if all $\mathbf{v}_i$ are unit length: $\|\mathbf{v}_i\| = 1$
they form an orthonormal basis

The *Gram-Schmidt method*:
— Basis of a linear space $\Rightarrow$ an orthonormal basis
— See the next Section

Inner Products

Cauchy-Schwartz inequality — in the context of inner product spaces

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \langle \mathbf{v}, \mathbf{v} \rangle \langle \mathbf{w}, \mathbf{w} \rangle$$

Equality holds if and only if **v** and **w** linearly dependent

Restate the Cauchy-Schwartz inequality

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

$$\left(\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} \right)^2 \leq 1$$

$$-1 \leq \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} \leq 1$$

Angle θ between **v** and **w**

$$\cos \theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

Inner Products

Inner product properties suggest

$$
\begin{aligned}
&\|\mathbf{v}\| \geq 0 \\
&\|\mathbf{v}\| = 0 \text{ if and only if } \mathbf{v} = 0 \\
&\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|
\end{aligned}
$$

A fourth property is the triangle inequality:

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

(derived from the Cauchy-Schwartz inequality in Chapter 2)

General definition of a projection

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ span a subspace $\mathcal{L}_k$ of $\mathcal{L}$
If $\mathbf{v}$ is a vector not in $\mathcal{L}_k$ then

$$P\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \ldots + \langle \mathbf{v}, \mathbf{u}_k \rangle \mathbf{u}_k$$

is $\mathbf{v}$'s orthogonal projection into $\mathcal{L}_k$

Gram-Schmidt Orthonormalization

Every inner product space has an orthonormal basis

Given: orthonormal vectors $\mathbf{b}_1, \ldots, \mathbf{b}_r$
— Form basis of subspace $\mathcal{S}_r$ of $\mathcal{L}_n$ where $n > r$
Find: b_{r+1} orthogonal to the given $\mathbf{b}_i$

Let $\mathbf{u}$ be an arbitrary vector in $\mathcal{L}_n$, but not in $\mathcal{S}_r$
$\mathbf{u}$'s *orthogonal projection* into $\mathcal{S}_r$:

$$\hat{\mathbf{u}} = \mathrm{proj}_{\mathcal{S}_r}\mathbf{u} = \langle \mathbf{u}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \ldots + \langle \mathbf{u}, \mathbf{b}_r \rangle \mathbf{b}_r$$

Check orthogonality: for example $\langle \mathbf{u} - \hat{\mathbf{u}}, \mathbf{b}_1 \rangle = 0$

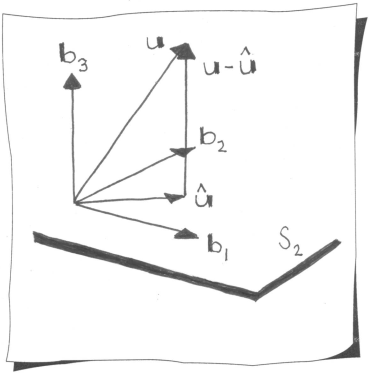
$$\langle \mathbf{u} - \hat{\mathbf{u}}, \mathbf{b}_1 \rangle = \langle \mathbf{u}, \mathbf{b}_1 \rangle - \langle \mathbf{u}, \mathbf{b}_1 \rangle \langle \mathbf{b}_1, \mathbf{b}_1 \rangle - \ldots - \langle \mathbf{u}, \mathbf{b}_r \rangle \langle \mathbf{b}_1, \mathbf{b}_r \rangle$$

$\Rightarrow$

$$\mathbf{b}_{r+1} = \frac{\mathbf{u} - \mathrm{proj}_{\mathcal{S}_r}\mathbf{u}}{\| \cdot \|}$$

Repeat to form an orthonormal basis for all of $\mathcal{L}_n$
Key elements: projections and vector decomposition

$\mathcal{S}_2$ is depicted as $\mathbb{R}^2$



Build the orthonormal basis:
Given basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathcal{L}_n$

$$\mathbf{b}_1 = \frac{\mathbf{v}_1}{\|\cdot\|}$$

$$\mathbf{b}_2 = \frac{\mathbf{v}_2 - \mathrm{proj}_{\mathcal{S}_1}\mathbf{v}_2}{\|\cdot\|} = \frac{\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{b}_1 \rangle \mathbf{b}_1}{\|\cdot\|}$$

$$\mathbf{b}_3 = \frac{\mathbf{v}_3 - \mathrm{proj}_{\mathcal{S}_2}\mathbf{v}_3}{\|\cdot\|}$$

$$= \frac{\mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{b}_1 \rangle \mathbf{b}_1 - \langle \mathbf{v}_3, \mathbf{b}_2 \rangle \mathbf{b}_2}{\|\cdot\|}$$

$$\vdots$$

Gram-Schmidt Orthonormalization

Example: $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

Form an orthonormal basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$

$$\mathbf{b}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix}0\\2/\sqrt{6}\\-1/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}$$

$$\mathbf{b}_4 = \frac{\mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{b}_1\rangle \mathbf{b}_1 - \langle \mathbf{v}_4, \mathbf{b}_2\rangle \mathbf{b}_2 - \langle \mathbf{v}_4, \mathbf{b}_3\rangle \mathbf{b}_3}{\|\cdot\|} = \begin{bmatrix}0\\0\\1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

Check: $\begin{vmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4\end{vmatrix} = 1$

A Gallery of Spaces

Let's highlight some special linear spaces—but there are many more!

A linear space $\mathcal{P}_n$ whose elements are all polynomials of a fixed degree n

$$p(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where t is the independent variable of $p(t)$
— Addition in this space is coefficient by coefficient
— Multiplication in this space: polynomial times a real number
Check linearity property: $p(t) = 3 - 2t + 3t^2$ and $q(t) = -1 + t + 2t^2$ then $2p(t) + 3q(t) = 3 - t + 12t^2$ is yet another polynomial of the same degree

$\Rightarrow$ *Linear map:* derivative p' of a degree n polynomial p

$$p'(t) = a_1 + 2a_2 t + \ldots + n a_n t^{n-1}$$

Rank of this map is $n - 1$

Example: Two cubic polynomials

$$p(t) = 3 - t + 2t^2 + 3t^3 \quad \text{and} \quad q(t) = 1 + t - t^3$$

in the linear space of cubic polynomials $\mathcal{P}_3$

$$\text{Let} \quad r(t) = 2p(t) - q(t) = 5 - 3t + 4t^2 + 7t^3$$

$$r'(t) = -3 + 8t + 21t^2$$
$$p'(t) = -1 + 4t + 9t^2$$
$$q'(t) = 1 - 3t^2$$

$r'(t) = 2p'(t) - q'(t) \Rightarrow$ linearity of the derivative map

A Gallery of Spaces

A linear space given by the set of all real-valued continuous functions over the interval $[0, 1]$
— This space is typically named $C[0, 1]$
— The linearity condition is met:
  If f and g are elements of $C[0, 1]$ then $\alpha f + \beta g$ is also in $C[0, 1]$
— This is an *infinite-dimensional* linear space
  No finite set of functions forms a basis for $C[0, 1]$

The set of all 3×3 matrices form a linear space
— This space consists of matrices
— Linear combinations formed using standard matrix addition and multiplication with a scalar

A Gallery of Spaces

A more abstract example:
The linear space formed from
the set of all linear maps from a linear space $\mathcal{L}_n$ into the reals
— Called the dual space $\mathcal{L}_n^*$ of $\mathcal{L}_n$
— Its dimension equals that of $\mathcal{L}_n$
— The linear maps in $\mathcal{L}_n^*$ are known as linear functionals

Let a fixed vector $\mathbf{v}$ and an variable vector $\mathbf{u}$ be in $\mathcal{L}_n$
The linear functionals defined by $\Phi_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$ are in $\mathcal{L}_n^*$
For any basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ of $\mathcal{L}_n$ define linear functionals

$$\Phi_{\mathbf{b}_i}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{b}_i \rangle \quad \text{for } i = 1, \ldots, n$$

These functionals form a basis for $\mathcal{L}_n^*$

Example: In $\mathbb{R}^2$ consider the fixed vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Then $\Phi_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle = u_1 - 2u_2$ for all vectors $\mathbf{u}$
where $\langle \cdot, \cdot \rangle$ is the dot product

Example: Pick $\mathbf{e}_1, \mathbf{e}_2$ for a basis in $\mathbb{R}^2$
The associated linear functionals are

$$\Phi_{\mathbf{e}_1}(\mathbf{u}) = u_1, \quad \Phi_{\mathbf{e}_2}(\mathbf{u}) = u_2$$

Any linear functional Φ can now be defined as

$$\Phi(\mathbf{u}) = r_1 \Phi_{\mathbf{e}_1}(\mathbf{u}) + r_2 \Phi_{\mathbf{e}_2}(\mathbf{u})$$

where r_1 and r_2 are scalars

WYSK

- linear space
- vector space
- dimension
- linear combination
- linearity property
- linearly independent
- subspace
- span
- linear map
- image
- preimage
- domain
- range
- rank
- full rank
- rank deficient
- inverse
- determinant
- inner product
- inner product space
- distance in an inner product space
- length in an inner product space
- orthogonal
- Gram-Schmidt method
- projection
- basis
- orthonormal
- orthogonal decomposition
- best approximation
- dual space
- linear functional