Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 19: Conics

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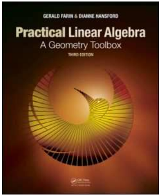
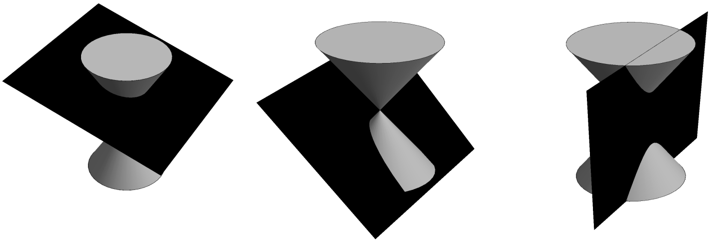
www.farinhansford.com/books/pla



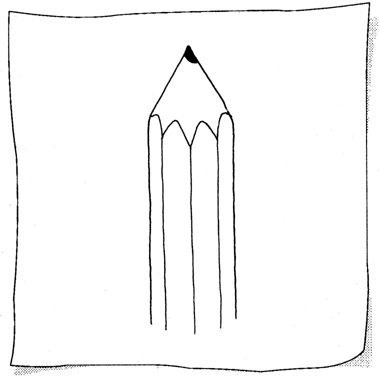
Outline

Introduction to Conics

Left to right: ellipse, parabola, and hyperbola



Take a flashlight and shine it straight onto a wall ⇒ circle
Tilt the light ⇒ ellipse
Tilt further ⇒ parabola ⇒ one branch of a hyperbola
Flashlight beam is a *cone* and wall is a plane
⇒ cone-plane intersection ⇒ conic section

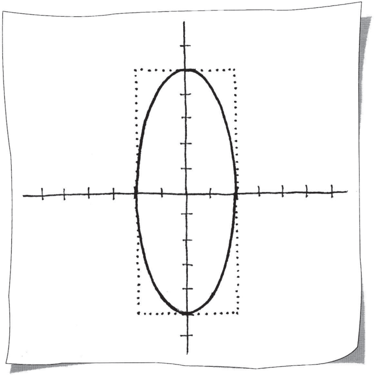


Some "real-life" occurrences of conic sections

- The paths of the planets around the sun are ellipses
- If you sharpen a pencil you generate hyperbolas
- If you water your lawn the water leaving the hose traces a parabolic arc

The General Conic

An ellipse: $\frac{1}{4}x_1^2 + \frac{1}{25}x_2^2 = 1$



Circle: $x_1^2 + x_2^2 = r^2$

Radius r and centered at the origin

Ellipse: $\lambda_1 x_1^2 + \lambda_2 x_2^2 = c$

In standard position

— Minor axis and major axis coincident with coordinate axes

— Center at the origin

— x_1 extents: $[-\sqrt{c/\lambda_1}, \sqrt{c/\lambda_1}]$

— x_2 extents: $[-\sqrt{c/\lambda_2}, \sqrt{c/\lambda_2}]$

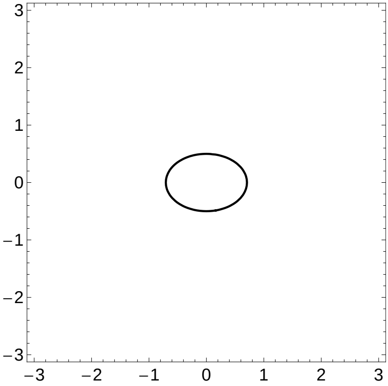
The General Conic

Rewrite ellipse in matrix form:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - c = 0$$

$$\mathbf{x}^{\mathrm{T}} D \mathbf{x} - c = 0$$

The General Conic



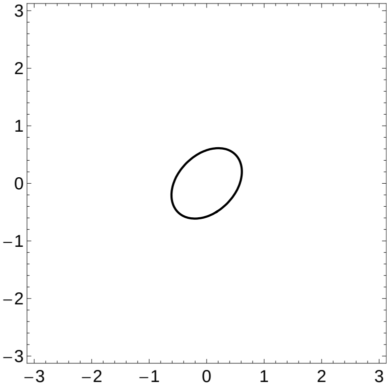
Example: Ellipse $2x_1^2 + 4x_2^2 - 1 = 0$
In matrix form

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 = 0$$

Major axis: on the $\mathbf{e}_1$-axis with extents $[-1/\sqrt{2}, 1/\sqrt{2}]$

Minor axis: on the $\mathbf{e}_2$-axis with extents $[-1/2, 1/2]$

Ellipse rotated out of standard position



Point $\hat{\mathbf{x}}$ on this ellipse

$\hat{\mathbf{x}} = R\mathbf{x} \quad \Rightarrow \quad \mathbf{x} = R^{\mathrm{T}}\hat{\mathbf{x}}$

$$[R^{\mathrm{T}}\hat{\mathbf{x}}]^T D[R^{\mathrm{T}}\hat{\mathbf{x}}] - c = 0$$

$$\hat{\mathbf{x}}^{\mathrm{T}} RDR^{\mathrm{T}}\hat{\mathbf{x}} - c = 0$$

$$A = RDR^{\mathrm{T}} \qquad (*)$$

$$\hat{\mathbf{x}}^{\mathrm{T}} A\hat{\mathbf{x}} - c = 0$$

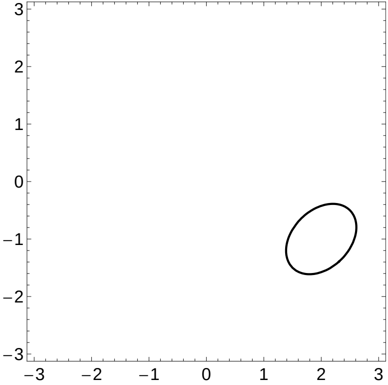
— Contour of a quadratic form

— A is a symmetric matrix

— (*) is eigendecompostion of A

The General Conic

Ellipse rotated and translated out of standard position



Point $\hat{\mathbf{x}}$ on this ellipse

$\hat{\mathbf{x}} = R\mathbf{x} + \mathbf{v} \quad \Rightarrow \quad \mathbf{x} = R^{\mathrm{T}}(\hat{\mathbf{x}} - \mathbf{v})$

$$[R^{\mathrm{T}}(\hat{\mathbf{x}} - \mathbf{v})]^T D[R^{\mathrm{T}}(\hat{\mathbf{x}} - \mathbf{v})] - c = 0$$

$$[\hat{\mathbf{x}}^{\mathrm{T}} - \mathbf{v}^{\mathrm{T}}]RDR^{\mathrm{T}}[\hat{\mathbf{x}} - \mathbf{v}] - c = 0$$

$$[\hat{\mathbf{x}}^{\mathrm{T}} - \mathbf{v}^{\mathrm{T}}]A[\hat{\mathbf{x}} - \mathbf{v}] - c = 0$$

The General Conic

Rotated and translated ellipse:

$$[\hat{\mathbf{x}}^{\mathrm{T}} - \mathbf{v}^{\mathrm{T}}]A[\hat{\mathbf{x}} - \mathbf{v}] - c = 0 \qquad \text{(drop "hat" notation)}$$

Symmetry of $A \quad \Rightarrow \quad \mathbf{x}^{\mathrm{T}}A\mathbf{v} = \mathbf{v}^{\mathrm{T}}A\mathbf{x}$

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}A\mathbf{v} + \mathbf{v}^{\mathrm{T}}A\mathbf{v} - c = 0$$

Abbreviate with $\mathbf{b} = A\mathbf{v}$ and $d = \mathbf{v}^{\mathrm{T}}A\mathbf{v} - c$

$$\text{Ellipse in general position:} \quad \mathbf{x}^{\mathrm{T}}A\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\mathbf{b} + d = 0$$

Can be written as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & \frac{1}{2}c_3 \\ \frac{1}{2}c_3 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 2 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}c_4 \\ -\frac{1}{2}c_5 \end{bmatrix} + c_6 = 0$$

Leads to the familiar equation of a conic

$$c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 + c_4x_1 + c_5x_2 + c_6 = 0$$

The General Conic

Example: Ellipse in standard position: $2x_1^2 + 4x_2^2 - 1 = 0$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 = 0$$

Rotate by 45° using the rotation matrix

$$R = \begin{bmatrix} s & -s \\ s & s \end{bmatrix} \qquad s = \sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$$

$A = RDR^{\mathrm{T}}$ becomes

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

Expanding: $3x_1^2 - 2x_1x_2 + 3x_2^2 - 1 = 0$

The General Conic

Example: Translate by

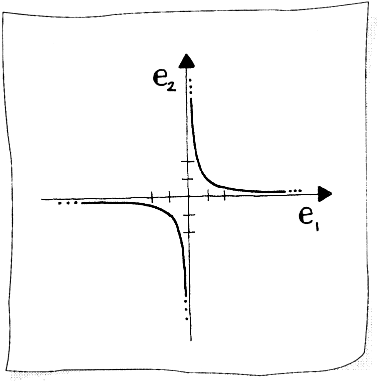
$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

the ellipse is now

$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{x} - 2\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 7 \\ -5 \end{bmatrix} + 18 = 0$$

Expanding: $3x_1^2 - 2x_1x_2 + 3x_2^2 - 14x_1 + 10x_2 + 18 = 0$

The General Conic



Any conic is represented by

$$\mathbf{x}^{\mathrm{T}} A\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\mathbf{b} + d = 0$$

Example: Hyperbola

$$x_1 x_2 - 1 = 0$$

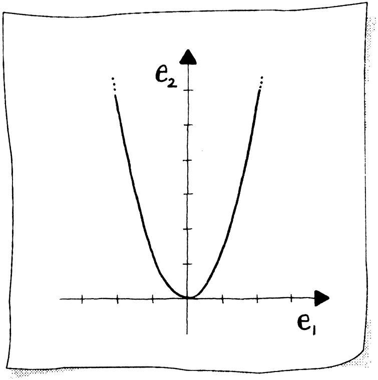
or the more familiar form

$$x_2 = \frac{1}{x_1}$$

In matrix form

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - 1 = 0$$

The General Conic



Example: Parabola

$$x_1^2 - x_2 = 0$$

or

$$x_2 = x_1^2$$

In matrix form

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

Analyzing Conics

Given: a conic $c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 + c_4x_1 + c_5x_2 + c_6 = 0$

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\mathbf{b} + d = 0$$

Find: the conic type

Solution:
- $\det A > 0 \Rightarrow$ ellipse
- $\det A = 0 \Rightarrow$ parabola
- $\det A < 0 \Rightarrow$ hyperbola

If $A =$ zero matrix and c_4 or c_5 non-zero $\Rightarrow$ straight line

Eigendecomposition $A = RDR^{\mathrm{T}}$ — eigenvalues are D's diagonal elements
- Two nonzero entries of the same sign: ellipse
- One nonzero entry: parabola
- Two nonzero entries with opposite sign: hyperbola

These conditions are summarized by $\det D$
— Rotations do not change areas $\Rightarrow$ checking A suffices

Analyzing Conics

Revisit previous examples

Example: An ellipse in two forms — in standard position and rotated

$$\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8$$

Determinant is positive ⇒ ellipse

Example: Hyperbola

$$\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4}$$

Determinant is negative ⇒ hyperbola

The characteristic equation is $(\lambda + 1/2)(\lambda - 1/2) = 0$
— eigenvalues have opposite sign ⇒ hyperbola

Analyzing Conics

Example: Parabola

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Determinant is zero $\Rightarrow$ parbola

The characteristic equation is $\lambda(\lambda + 1) = 0$
— one eigenvalue is zero $\Rightarrow$ parabola

The conic type is determined by the sign of the determinant of A
— it is unchanged by (invertible) affine maps

General Conic to Standard Position

Given: $c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 + c_4x_1 + c_5x_2 + c_6 = 0$

Find: conic's equation in standard position
— Degree of freedom: major axis can coincide with $\mathbf{e}_1$- or $\mathbf{e}_2$-axis

Recall matrix form

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\mathbf{b} + d = 0 \quad \text{where} \quad \mathbf{b} = A\mathbf{v} \quad \text{and} \quad d = \mathbf{v}^{\mathrm{T}}A\mathbf{v} - c$$

Demonstrate the solution with ellipse from previous slides:

$$3x_1^2 - 2x_1x_2 + 3x_2^2 - 14x_1 + 10x_2 + 18 = 0$$

$$\text{Matrix form:} \quad \mathbf{x}^{\mathrm{T}}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\mathbf{x} - 2\mathbf{x}^{\mathrm{T}}\begin{bmatrix} 7 \\ -5 \end{bmatrix} + 18 = 0$$

Find translation $\mathbf{v}$ found by solving $A\mathbf{v} = \mathbf{b} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Can solve $A\mathbf{v} = \mathbf{b}$ if A full rank
$\Rightarrow$ two nonzero eigenvalues $\Rightarrow$ ellipse or hyperbola

General Conic to Standard Position

Next: remove the translation
Calculate $c = \mathbf{v}^T A\mathbf{v} - d = 1 \quad \Rightarrow \quad$ ellipse with center at the origin

$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{x} - 1 = 0$$

Characteristic equation:

$$\lambda^2 - 6\lambda + 8 = 0 \quad \Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 2$$

Resulting in

$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x} - 1 = 0$$

(Major axis aligned with the $\mathbf{e}_2$-axis)

General Conic to Standard Position

Next: find the rotation

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad \lambda_1 = 4, \quad \lambda_2 = 2$$

leads to the eigenvectors that form the columns of

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

This is a -45° rotation

Example: Find the type and equation in standard position of

$$x_1^2 + 2x_2^2 + 8x_1x_2 - 4x_1 - 16x_2 + 3 = 0$$

Matrix form: $$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{x} - 2\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 2 \\ 8 \end{bmatrix} + 3 = 0$$

Matrix determinant negative $\Rightarrow$ hyperbola
Recover the translation

$$\begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Calculate $c = \mathbf{v}^{\mathrm{T}} A \mathbf{v} - 3 = 1 \quad \Rightarrow$ conic without the translation

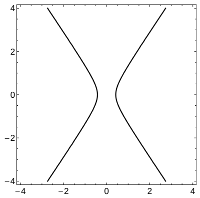
$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{x} - 1 = 0$$

General Conic to Standard Position

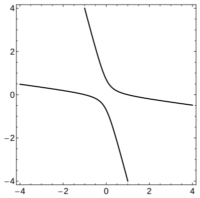
$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 1 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{x} - 1 = 0$$

Characteristic equation: $\lambda^2 - 3\lambda - 14 = 0 \Rightarrow \lambda_1 = 5.53$ and $\lambda_2 = -2.53$
The hyperbola in standard position

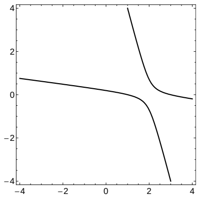
$$\mathbf{x}^{\mathrm{T}} \begin{bmatrix} 5.53 & 0 \\ 0 & -2.53 \end{bmatrix} \mathbf{x} - 1 = 0$$



standard position



with rotation



with rotation and translation

WYSK

- conic section
- implicit equation
- circle
- quadratic form
- ellipse
- minor axis
- semi-minor axis
- major axis
- semi-major axis
- center
- standard position
- conic type
- hyperbola
- parabola
- straight line
- eigenvalues
- eigenvectors
- eigendecomposition
- affine invariance