Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 2: Here and There: Points and Vectors in 2D

Gerald Farin & Dianne Hansford

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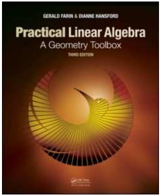
www.farinhansford.com/books/pla



Outline

Hurricane Katrina approaching south Louisiana
Air is moving rapidly – spiraling counterclockwise
Moving faster as it approaches the eye of the hurricane

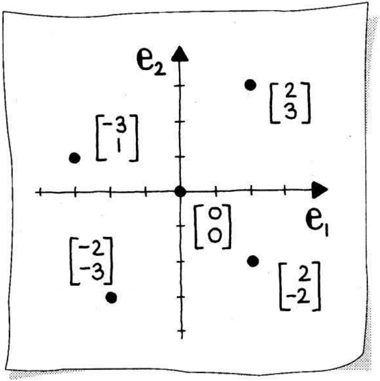
Air movement can be described by

- points: location
- vectors: direction and speed

2D slices – cross sections – provide depth information

This chapter introduces points and vectors in 2D

Points and Vectors



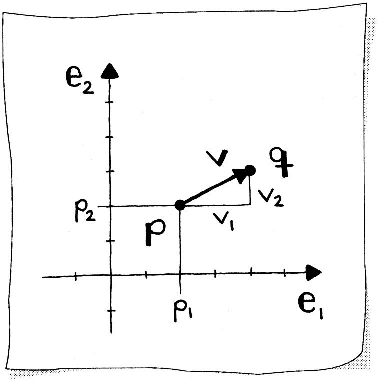
Point: reference to a location
Notation: boldface lowercase letters

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

p_1 and p_2 are coordinates

2D points "live" in
2D Euclidean space $\mathbb{E}^2$

Points and Vectors



Vector: difference of two points
Notation: boldface lowercase letters

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

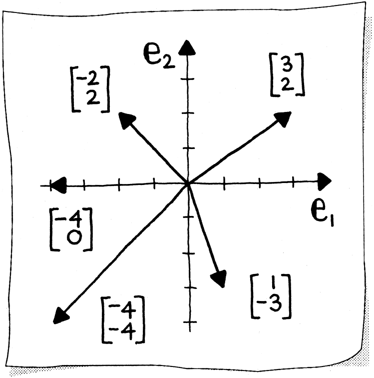
v_1 and v_2 are components
Move from **p** to **q**:

$$\mathbf{q} = \mathbf{p} + \mathbf{v}$$

Calculate each component separately

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} p_1 + v_1 \\ p_2 + v_2 \end{bmatrix}$$

Points and Vectors

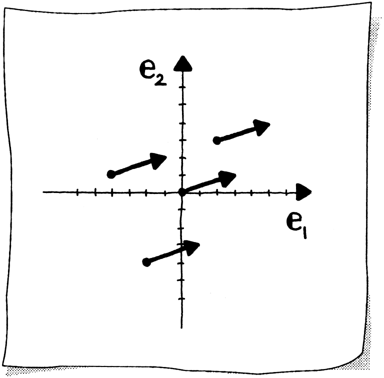


Vector: direction and distance (displacement)

$$\mathbf{v} = \mathbf{q} - \mathbf{p}$$

Length can be interpreted in variety of ways
Examples: distance, speed, force

Points and Vectors



Vector has a *tail* and a *head*

Unlike a point, a vector does *not* define a position

Two vectors are equal if have the same component values

Any number of vectors have same direction and length

Points and Vectors

Special examples:

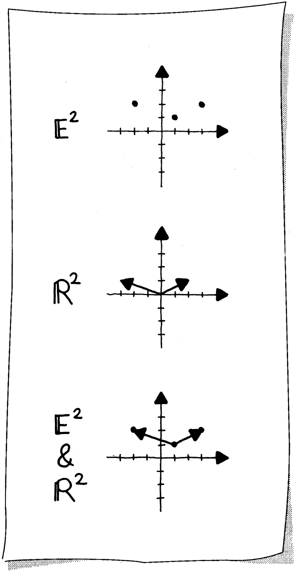
Zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This vector has no direction or length

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2D vectors "live" in 2D linear space $\mathbb{R}^2$

Other names for $\mathbb{R}^2$: real space or vector spaces

What's the Difference?



Notation and data structure for points and vectors the same

Can they be used interchangeably?

No!

Point lives in $\mathbb{E}^2$

Vector lives in $\mathbb{R}^2$

Euclidean and linear spaces illustrated separately and together.

What's the Difference?

Primary reason for differentiating between points and vectors:
achieve geometric constructions that are coordinate independent

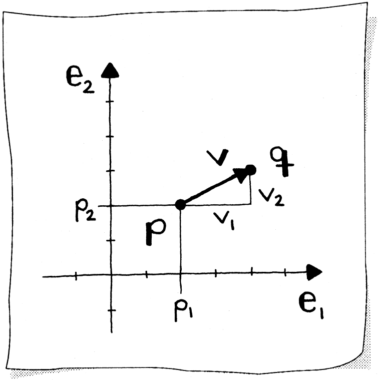
$\Rightarrow$ Manipulations applied to geometric objects produce the same result regardless of location of the coordinate origin

Example: the midpoint of two points

Idea becomes clearer by analyzing some fundamental manipulations of points and vectors

Let $\mathbf{p}, \mathbf{q} \in \mathbb{E}^2$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$

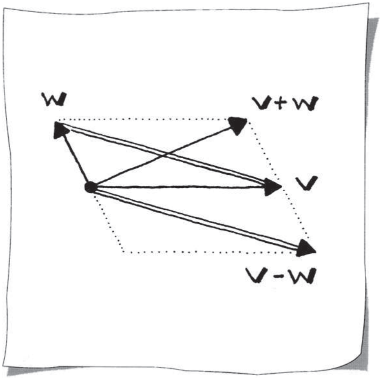
Coordinate Independent Operation



Subtracting a point from another point:
$(\mathbf{q} - \mathbf{p})$ yields a vector $\mathbf{v}$

What's the Difference?

Coordinate Independent Operation



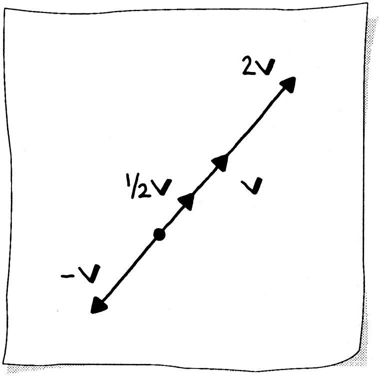
Adding or subtracting two vectors yields another vector

Parallelogram rule:

Vectors $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are the diagonals of the parallelogram defined by $\mathbf{v}$ and $\mathbf{w}$

This is a coordinate independent operation since vectors are defined as a difference of points

Coordinate Independent Operation



Scaling: multiplying a vector by a scalar s

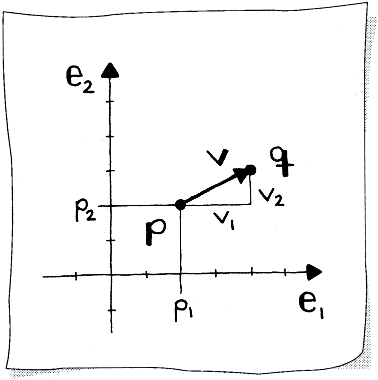
Scaling a vector is a well-defined operation

Result $s\mathbf{v}$ adjusts the length by the scaling factor

Direction unchanged if $s > 0$
Direction reversed for $s < 0$
If $s = 0$ result is the zero vector

What's the Difference?

Coordinate Independent Operation



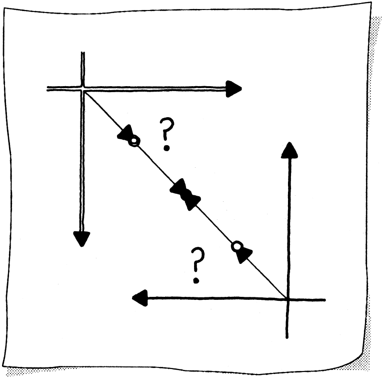
Adding a vector to a point ($\mathbf{p} + \mathbf{v}$) yields another point

Any coordinate independent combination of two or more points and/or vectors is formed from one or more of the coordinate independent operations

Example: $\mathbf{p} + \frac{1}{2}\mathbf{w}$

What's the Difference?

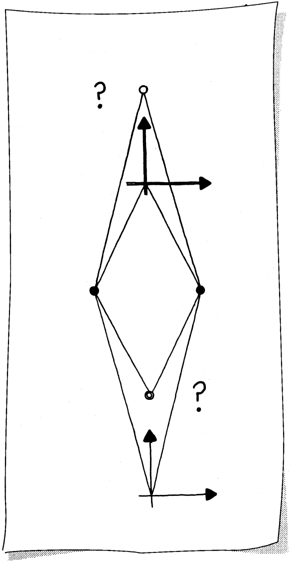
Coordinate Dependent Operation



Scaling a point ($s\mathbf{p}$) is not a well-defined operation because it is not coordinate independent

Scaling the solid black point by one-half with respect to two different coordinate systems results in two different points

What's the Difference?



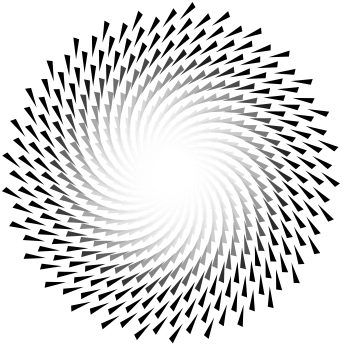
Coordinate Dependent Operation

Adding two points ($\mathbf{p} + \mathbf{q}$) is not a well-defined operation

Result of adding the two solid black points is dependent on the coordinate origin
(Parallelogram rule used here to construct the results of the additions)

Some special combinations of points are allowed – more on that later

Vector Fields

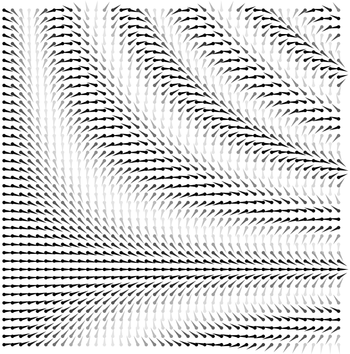


Vector field: every point in a given region is assigned a vector

Example: simulating air velocity – lighter gray indicates greater velocity

Visualization of a vector field requires *discretizing* it: finite number of point and vector pairs selected

Vector Fields



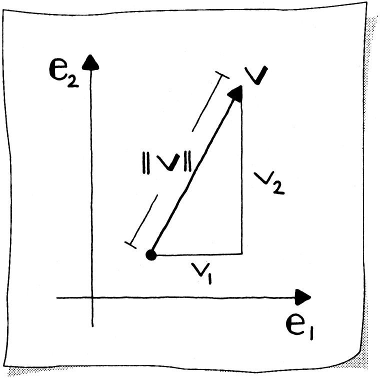
Other important applications of vector fields: automotive and aerospace design

Before a car or an airplane is built, it undergoes extensive aerodynamic simulations

In these simulations, the vectors that characterize the flow around an object are computed from complex differential equations

Length of a Vector

Length or magnitude of a vector can represent distance, velocity, or acceleration



Denote length of $\mathbf{v}$ as $\|\mathbf{v}\|$

Pythagorean theorem:

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

Euclidean norm of a vector

Length of a Vector

Scale the vector by an amount k then $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

Normalized vector $\mathbf{w}$ has unit length: $\|\mathbf{w}\| = 1$
Normalized vectors also known as unit vectors

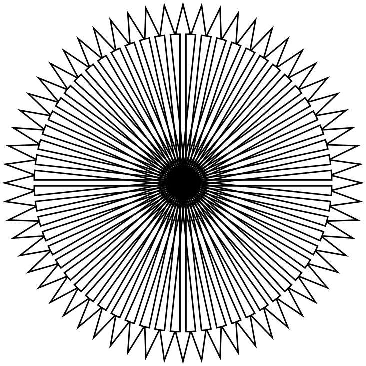
Normalize a vector: scale so that it has unit length
If $\mathbf{w}$ is unit length version of $\mathbf{v}$ then

$$\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Each component of $\mathbf{v}$ is divided by the *nonnegative* scalar value $\|\mathbf{v}\|$

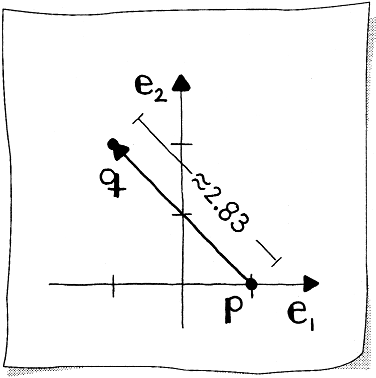
Length of a Vector

Unit vector: vector of length 1



There are infinitely many unit vectors.
Imagine drawing them all ...
Resulting figure: a circle of radius one

Distance between two points



Form a vector defined by two points: $\mathbf{v} = \mathbf{q} - \mathbf{p}$
Then calculate $\|\mathbf{v}\|$

Example:

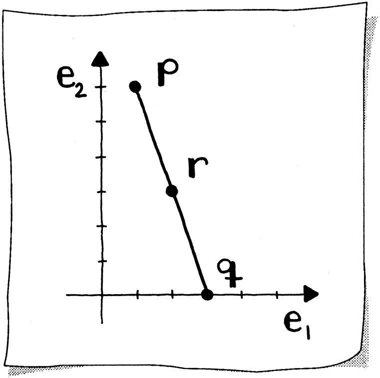
$$\mathbf{q} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{q} - \mathbf{p} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\|\mathbf{q} - \mathbf{p}\| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} \approx 2.83$$

Combining Points

Combine two points to get a (meaningful) third one



Example: form midpoint $\mathbf{r}$ of two points $\mathbf{p}$ and $\mathbf{q}$

$$\mathbf{p} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Combining Points

Construct midpoint using coordinate independent operations

Define $\mathbf{r}$ by adding an appropriately scaled version of vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$ to point $\mathbf{p}$:

$$\mathbf{r} = \mathbf{p} + \frac{1}{2}\mathbf{v}$$

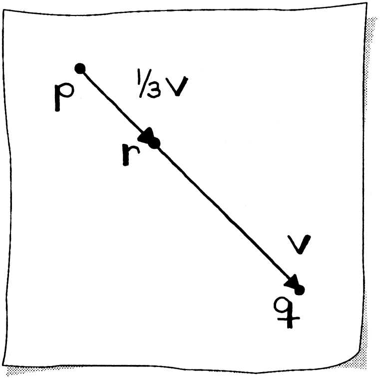
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$\mathbf{r}$ can also be defined as

$$\mathbf{r} = \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

This is a legal expression for a combination of points
Nothing magical about the factor 1/2 ...



Point $\mathbf{r} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ is on the line through $\mathbf{p}$ and $\mathbf{q}$

Equivalently

$$\mathbf{r} = (1 - t)\mathbf{p} + t\mathbf{q}$$

Sketch: $\mathbf{r} = \frac{2}{3}\mathbf{p} + \frac{1}{3}\mathbf{q}$

Scalar values $(1 - t)$ and t are coefficients

Combining Points

Barycentric combination: a weighted sum of points where the coefficients sum to one

$$\mathbf{r} = (1 - t)\mathbf{p} + t\mathbf{q}$$

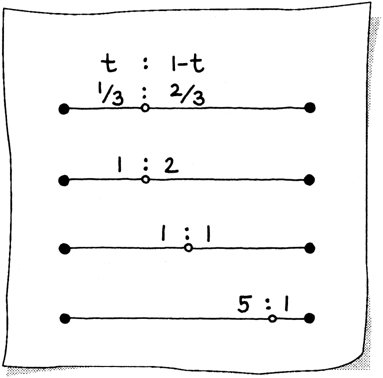
When one point **r** is expressed in terms of two others **p** and **q**:
coefficients $1 - t$ and t are called the barycentric coordinates of **r**

Can construct **r** anywhere on the line defined by **p** and **q**
Also called linear interpolation
In this context, t is called a parameter

If we restrict **r** to the *line segment* between **p** and **q**
then we allow only convex combinations: $0 \leq t \leq 1$

Define **r** outside of the line segment between **p** and **q**
then $t < 0$ or $t > 1$

Combining Points



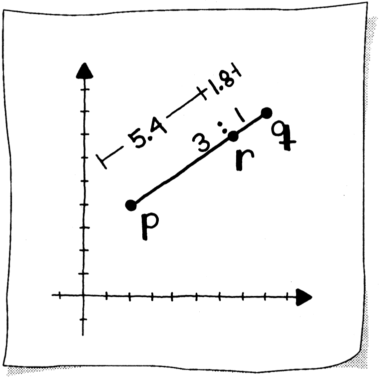
Position of **r** is in the ratio of
$t : (1 - t)$ or $t/(1 - t)$

$$\text{ratio} = \frac{||\mathbf{r} - \mathbf{p}||}{||\mathbf{q} - \mathbf{r}||}$$

In physics **r** is known as the *center of gravity* of **p** and **q** with weights $1 - t$ and t, resp.

Combining Points

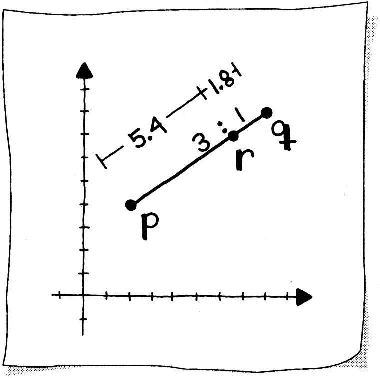
What are the barycentric coordinates of **r** with respect to **p** and **q**?



Ratio of **r** with respect to **p** and **q** is $s_1 : s_2$
Scale ratio values so that they sum to one –
resulting in $(1-t) : t$

$$t = \frac{||\mathbf{r}-\mathbf{p}||}{||\mathbf{q}-\mathbf{p}||}$$

Then $\mathbf{r} = (1-t)\mathbf{p} + t\mathbf{q}$



Example:

$$\begin{bmatrix} 6.5 \\ 7 \end{bmatrix} = (1-t)\begin{bmatrix} 2 \\ 4 \end{bmatrix} + t\begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

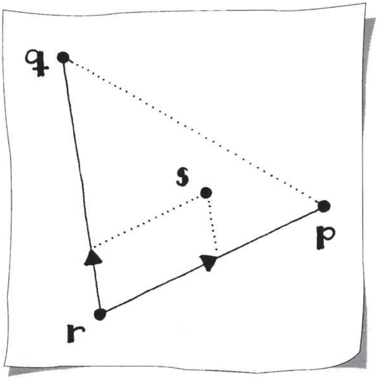
$$l_1 = \|\mathbf{r} - \mathbf{p}\| \approx 5.4$$
$$l_2 = \|\mathbf{q} - \mathbf{r}\| \approx 1.8$$
$$l_3 = l_1 + l_2 \approx 7.2$$

$$t = l_1/l_3 = 0.75$$
$$(1-t) = l_2/l_3 = 0.25$$

Verify:

$$\begin{bmatrix} 6.5 \\ 7 \end{bmatrix} = 0.25 \times \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 0.75 \times \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Barycentric combinations with more than two points



Given: three noncollinear points **p**, **q**, and **r**

Any point **s** can be formed from

$$\mathbf{s} = \mathbf{r} + t_1(\mathbf{p} - \mathbf{r}) + t_2(\mathbf{q} - \mathbf{r})$$

Coordinate independent operation:
point + vector + vector

$$\mathbf{s} = t_1\mathbf{p} + t_2\mathbf{q} + (1 - t_1 - t_2)\mathbf{r}$$
$$= t_1\mathbf{p} + t_2\mathbf{q} + t_3\mathbf{r}$$

t_1, t_2, t_3 are barycentric coordinates of **s** with respect to **p**, **q**, **r**

Combining Points

Combine points so result is a vector
$\Rightarrow$ Coefficients must sum to zero

Example:

$$\mathbf{e} = \mathbf{r} - 2\mathbf{p} + \mathbf{q}, \qquad \mathbf{r}, \mathbf{p}, \mathbf{q} \in \mathbb{E}^2$$

Does **e** have a geometric meaning?

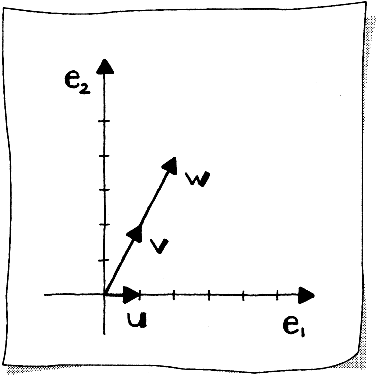
Sum of the coefficients: $1 - 2 + 1 = 0 \Rightarrow$ **e** is a vector

How to see this? Rewrite as

$$\mathbf{e} = (\mathbf{r} - \mathbf{p}) + (\mathbf{q} - \mathbf{p})$$

e is a vector formed from (vector + vector)

Independence



Two vectors **v** and **w** describe a parallelogram
If this parallelogram has zero area then the two vectors are parallel

$$\mathbf{v} = c\mathbf{w}$$

Vectors parallel
⇒ linearly dependent
Vectors not parallel
⇒ linearly independent

Independence

Two linearly independent vectors may be used to write any other vector $\mathbf{u}$ as a linear combination:

$$\mathbf{u} = r\mathbf{v} + s\mathbf{w}$$

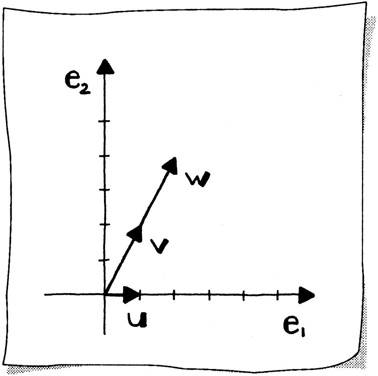
Two linearly independent vectors in 2D are also called a basis for $\mathbb{R}^2$

If $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent
then cannot write all vectors as a linear combination of them

Next: an example

Independence

Example:



Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Want to write

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{as} \quad \mathbf{u} = r\mathbf{v} + s\mathbf{w}$$

$$1 = r + 2s$$
$$0 = 2r + 4s$$

No r, s satisfies both equations
$\Rightarrow$ **u** *cannot* be written as a linear combination of **v** and **w**

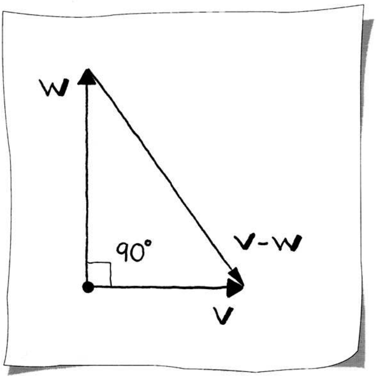
Dot Product

Given two vectors $\mathbf{v}$ and $\mathbf{w}$:

- Are they the *same* vector?
- Are they *perpendicular* to each other?
- What *angle* do they form?

The dot product resolves these questions

Dot Product



Two perpendicular vectors **v** and **w**
From the Pythagorean theorem

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

Expanding and bringing all terms to the left-hand side results in

$$v_1 w_1 + v_2 w_2 = 0 \quad \text{or} \quad \mathbf{v} \cdot \mathbf{w} = 0$$

Immediate application:

w perpendicular to **v** when

$$\mathbf{w} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

Dot Product

The dot product of arbitrary vectors $\mathbf{v}$ and $\mathbf{w}$ is

$$s = \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

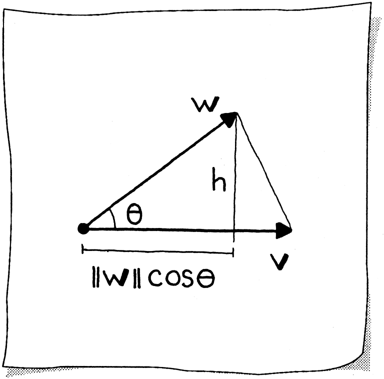
Dot product returns a scalar s
Called a scalar product or inner product

Symmetry property:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

Dot Product

Geometric meaning



From "left" triangle:

$$h^2 = \|\mathbf{w}\|^2(1 - \cos^2(\theta))$$

From "right" triangle:

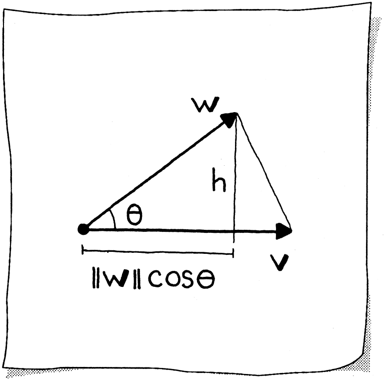
$$h^2 = \|\mathbf{v} - \mathbf{w}\|^2 - (\|\mathbf{v}\| - \|\mathbf{w}\| \cos\theta)^2$$

Equate ⇒ Law of Cosines

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

generalized Pythagorean theorem

Geometric meaning (con't)



Explicitly write $\|\mathbf{v} - \mathbf{w}\|^2$

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$$

Equating the expressions for $\|\mathbf{v} - \mathbf{w}\|^2$ results in:

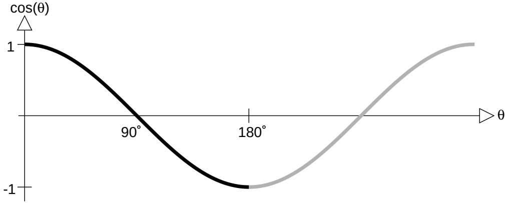
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

a very useful expression for the dot product

Dot Product

Cosine function:

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$ values range between ± 1

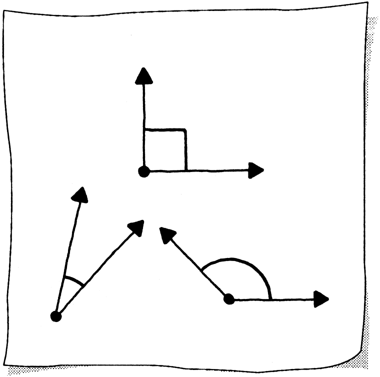


Perpendicular vectors: $\cos(90^\circ) = 0$

Same/opposite direction $\mathbf{v} = k\mathbf{w}$:

$$\cos\theta = \frac{k\mathbf{w}\cdot\mathbf{w}}{\|k\mathbf{w}\|\|\mathbf{w}\|} = \frac{k\|\mathbf{w}\|^2}{|k|\|\mathbf{w}\|\|\mathbf{w}\|} = \pm 1 \qquad \Rightarrow \qquad \theta = 0^\circ \text{ or } \theta = 180^\circ$$

Dot Product



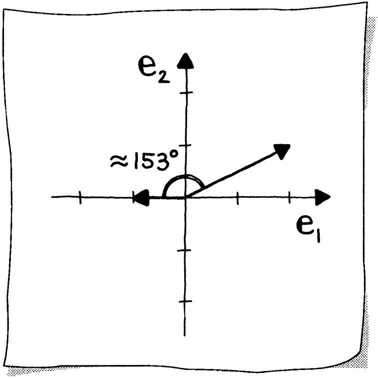
Three types of angles:

- *right:* $\cos(\theta) = 0 \Rightarrow \mathbf{v} \cdot \mathbf{w} = 0$
- *acute:* $\cos(\theta) > 0 \Rightarrow \mathbf{v} \cdot \mathbf{w} > 0$
- *obtuse:* $\cos(\theta) < 0 \Rightarrow \mathbf{v} \cdot \mathbf{w} < 0$

If θ needed
then use arccosine function

$$\theta = \mathrm{acos}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

Example:



$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Calculate the length of each vector

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\|\mathbf{w}\| = \sqrt{-1^2 + 0^2} = 1$$

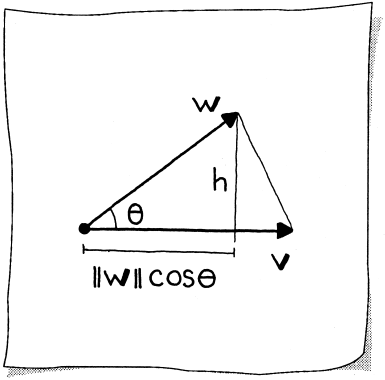
$$\cos(\theta) = \frac{(2 \times -1) + (1 \times 0)}{\sqrt{5} \times 1} \approx -0.8944$$

$$\arccos(-0.8944) \approx 153.4^\circ$$

Degrees to radians:

$$153.4^\circ \times \frac{\pi}{180^\circ} \approx 2.677 \text{ radians}$$

Orthogonal Projections

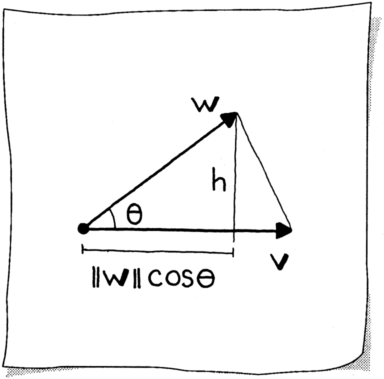


Projection of **w** onto **v** creates a footprint of length

$$b = ||\mathbf{w}|| \cos(\theta)$$

From basic trigonometry:
$\cos(\theta) = b/\text{hypotenuse}$ Orthogonal projection of **w** onto **v**:

$$\mathbf{u} = (||\mathbf{w}|| \cos(\theta)) \frac{\mathbf{v}}{||\mathbf{v}||}$$
$$= \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||^2} \mathbf{v}$$



$$\mathbf{u} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||^2}\mathbf{v} = \mathrm{proj}_{\mathcal{V}_1}\mathbf{w}$$

$\mathcal{V}_1$ is the set of all 2D vectors $k\mathbf{v}$
$\mathcal{V}_1$ is a 1D *subspace* of $\mathbb{R}^2$

u is the best approximation
to **w** in $\mathcal{V}_1$

Concept of best approximation
is important for many applications

Orthogonal Projections

Decompose $\mathbf{w}$ into a sum of two perpendicular vectors:

$$\mathbf{w} = \mathbf{u} + \mathbf{u}^{\perp}$$

$\mathbf{w}$ *resolved* into components with respect to two other vectors

$$\mathbf{u}^{\perp} = \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||^2}\mathbf{v}$$

This can also be written as

$$\mathbf{u}^{\perp} = \mathbf{w} - \text{proj}_{\mathcal{V}_1}\mathbf{w}$$

$\mathbf{u}^{\perp}$ is component of $\mathbf{w}$ orthogonal to the space of $\mathbf{u}$

Orthogonal projections and vector decomposition are at the core of constructing the orthonormal coordinate frames

Vector decompostion is key to Fourier analysis, quantum mechanics, digital audio, video recording

Inequalities

Cauchy-Schwartz Inequality

$$(\mathbf{v} \cdot \mathbf{w})^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

Derived from:

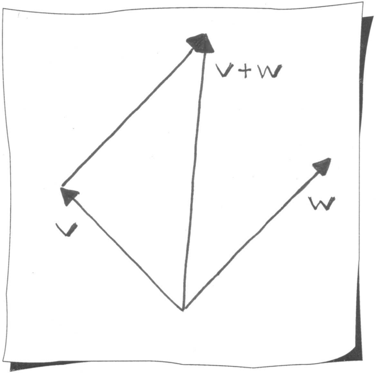
$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$

Squaring both sides

$$(\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \cos^2\theta$$

and note that $0 \leq \cos^2\theta \leq 1$

Inequalities



The triangle inequality

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

WYSK

- point versus vector
- coordinates versus components
- $\mathbb{E}^2$ versus $\mathbb{R}^2$
- coordinate independent
- vector length
- unit vector
- zero divide tolerance
- Pythagorean theorem
- distance between two points
- parallelogram rule
- scaling
- ratio
- barycentric combination
- linear interpolation
- convex combination
- barycentric coordinates
- linearly dependent vectors
- linear combination
- basis for $\mathbb{R}^2$
- dot product
- Law of Cosines
- perpendicular vectors
- angle between vectors
- orthogonal projection
- vector decomposition
- Cauchy-Schwartz inequality
- triangle inequality