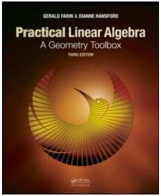
Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 3: Lining Up: 2D Lines

Gerald Farin & Dianne Hansford

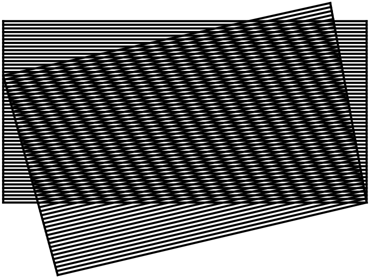
CRC Press, Taylor & Francis Group, An A K Peters Book
www.farinhansford.com/books/pla



Outline

Introduction to 2D Lines

2D lines are the building blocks
for many geometric constructions



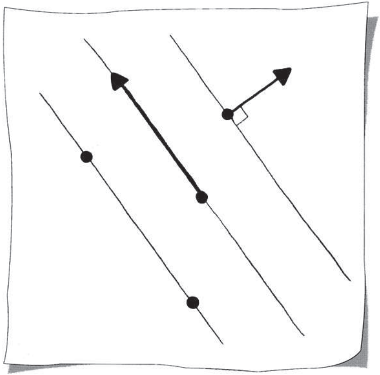
Two sets of parallel lines overlaid
Interference pattern called
Moiré pattern
Used in optics for checking the
properties of lenses

Chapter focus:

- representations for lines
- distance from a line
- intersections

Defining a Line

Two elements of 2D geometry define a line:



Elements can be

- two points
- a point and a vector parallel to the line
- a point and a vector perpendicular to the line

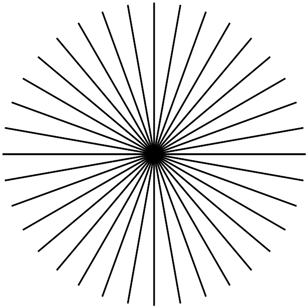
Normal to a line: unit vector that is perpendicular (or orthogonal) to a line

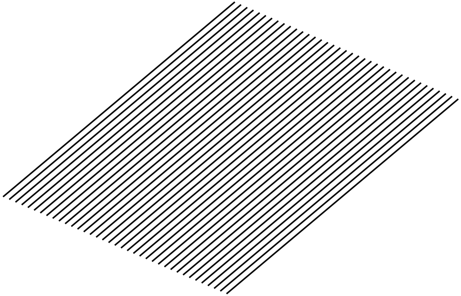
Defining a Line

Families of lines

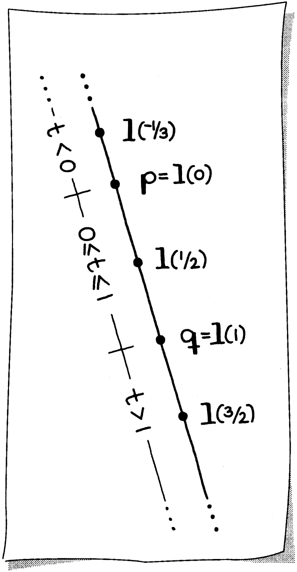
One family of lines shares a common point

The other family of lines shares the same normal





Parametric Equation of a Line



$$\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$$

where $\mathbf{p} \in \mathbb{E}^2$ and $\mathbf{v} \in \mathbb{R}^2$

Scalar value t is the parameter

Evaluating for specific parameter generates a point on the line

Parametric Equation of a Line

Parametric form $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$
in terms of barycentric coordinates:

Let $\mathbf{v} = \mathbf{q} - \mathbf{p}$, then

$$\mathbf{l}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$$

$(1 - t)$ and t are the barycentric coordinates
of a point on the line with respect to $\mathbf{p}$ and $\mathbf{q}$, resp.

Typically referred to as linear interpolation

Parametric Equation of a Line

Barycentric combination: sum of coefficients of points is one

$$\mathbf{l}(t) = (1-t)\mathbf{p} + t\mathbf{q}$$

For $t \in [0,1]$: generates points on line between $\mathbf{p}$ and $\mathbf{q}$

This is a convex combination

$t < 0$ generates points on the line "behind" $\mathbf{p}$
$t > 1$ generates points "past" $\mathbf{q}$
(Interpret as scaling of $\mathbf{v}$ in $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$)
This is extrapolation

Parametric Equation of a Line

Parametric form very good for computing points on a line

Example: compute ten equally spaced points on line segment through **p** and **q**

$$\mathbf{l}(t) = (1-t)\mathbf{p} + t\mathbf{q}$$

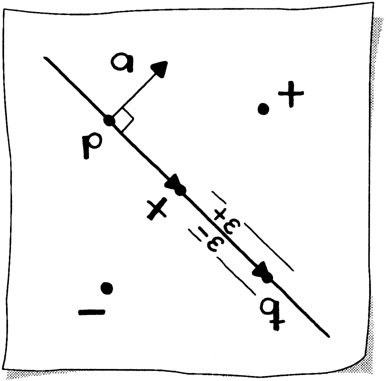
$$t = i/9, \qquad i = 0, \ldots, 9$$

$$\begin{aligned} i = 0 \quad &\rightarrow \quad t = 0 \quad \rightarrow \quad \mathbf{p} \\ i = 9 \quad &\rightarrow \quad t = 1 \quad \rightarrow \quad \mathbf{q} \end{aligned}$$

Equally spaced parameter values correspond to equally spaced points

Parametrization: speed line is traversed

Implicit Equation of a Line



Point **p** and a vector **a** perpendicular to the line For any point **x** on the line

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

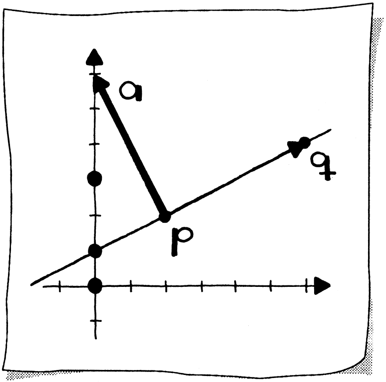
a and (**x** − **p**) are perpendicular

If **a** unit length,
called point normal form

Expand:
$a_1 x_1 + a_2 x_2 + (-a_1 p_1 - a_2 p_2) = 0$
Results in familiar form:

$$a x_1 + b x_2 + c = 0$$

Implicit Equation of a Line



Given: two points on the line

$$\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Find: implicit line $ax_1 + bx_2 + c = 0$

$$\mathbf{v} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$c = 2 \times 2 - 4 \times 2 = -4$

Implicit equation of the line:

$-2x_1 + 4x_2 - 4 = 0$

(Not point normal form)

Implicit Equation of a Line

Implicit form good for deciding if an arbitrary point lies on the line

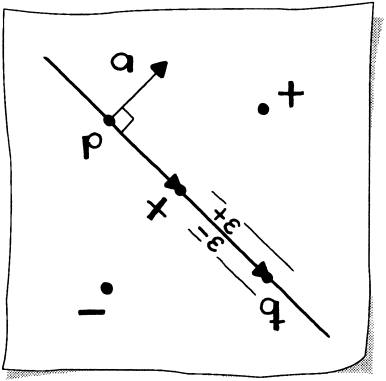
For given point $\mathbf{x}$

$$f = ax_1 + bx_2 + c$$

If $f = 0$ then the point is on the line

Implicit Equation of a Line

Numerical caveat:
Checking equality $f = 0.0$ with floating point numbers not recommended



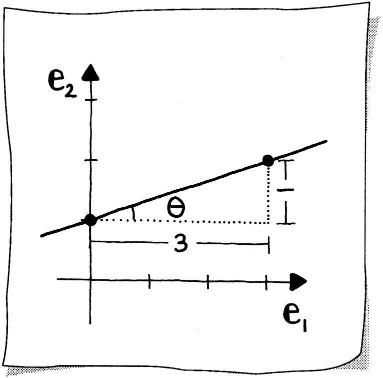
Tolerance ϵ needed
Meaningful tolerance:
true distance of $\mathbf{x}$ to line

$$d = \frac{f}{\|\mathbf{a}\|}$$

Sign of d indicates side of the line

Explicit Equation of a Line

Explicit form is closely related to the implicit form $ax_1 + bx_2 + c = 0$



Expresses x_2 as a function of x_1

$$x_2 = -\frac{a}{b}x_1 - \frac{c}{b}$$

Simpler: $x_2 = \hat{a}x_1 + \hat{b}$

Geometric meaning of coefficients:
$\hat{a}$ is the slope   "rise/run"
$\hat{b}$ is the $\mathbf{e}_2$-intercept
Example: $x_2 = 1/3x_1 + 1$

Drawback: vertical line has infinite slope

Converting Between Parametric and Implicit Equations

Advantages to both the parametric and implicit representations of a line

Depending on the geometric algorithm
may be convenient to use one form rather than the other

Ignore the explicit form:
not very useful for general 2D geometry

Converting Between Parametric and Implicit Equations

Parametric to Implicit

Given: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$
Find: coefficients a, b, c of $ax_1 + bx_2 + c = 0$
Solution: Form a vector $\mathbf{a} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$ perpendicular to vector $\mathbf{v}$
Determines the coefficients $a = a_1$ and $b = a_2$
Use $\mathbf{p}$ from $\mathbf{l}(t)$ to compute $c = -(a_1 p_1 + a_2 p_2)$

Converting Between Parametric and Implicit Equations

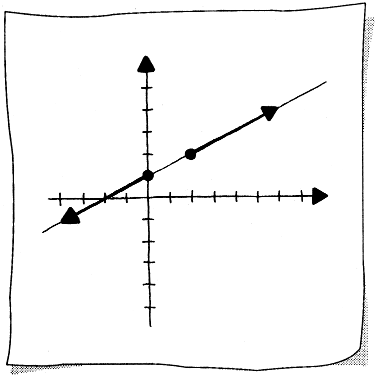
Implicit to Parametric

Given: $ax_1 + bx_2 + c = 0$
Find: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$
Solution: Need one point on the line and a vector parallel to the line
Vector $\mathbf{v} = \begin{bmatrix} b \\ -a \end{bmatrix}$ is perpendicular to $\mathbf{a}$
Point candidates: intersections with the $\mathbf{e}_1$- or $\mathbf{e}_2$-axis

$$\begin{bmatrix} -c/a \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ -c/b \end{bmatrix}$$

For numerical stability, choose the intersection closest to the origin

Non-uniqueness of representations



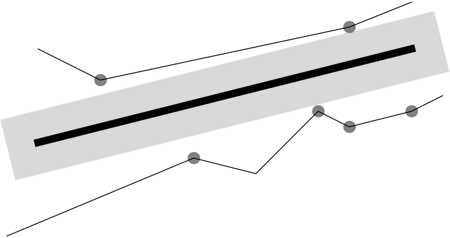
Two parametric representations for the same line
Lines will be *traced* will differently

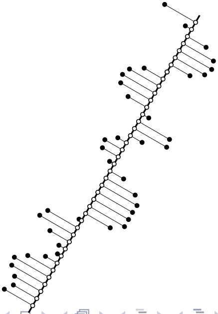
Conversion process – for example
parametric $\rightarrow$ implicit $\rightarrow$ parametric
Original and final parametric form different in general

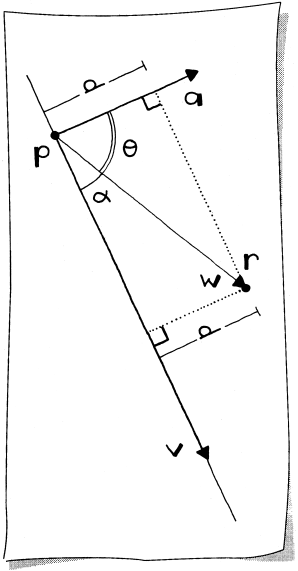
Distance of a Point to a Line

Robot will travel along the line (thick black)
Points represent objects in the room
Verify needed robot clearance (gray)
by computing distance of point to line

Smallest distance $d(\mathbf{r}, \mathbf{l})$
of a point to a line is the
orthogonal or perpendicular distance







Starting with an Implicit Line

Given: $\mathbf{l} : ax_1 + bx + c$ and point $\mathbf{r}$
Find: $d(\mathbf{r}, \mathbf{l})$
Solution: let $\mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix}$

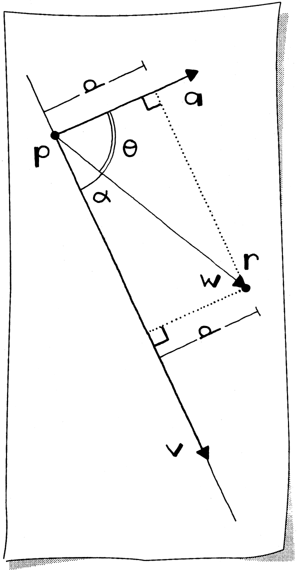
$$d = \frac{ar_1 + br_2 + c}{\|\mathbf{a}\|} = \frac{\mathbf{a} \cdot (\mathbf{r} - \mathbf{p})}{\|\mathbf{a}\|}$$

Let $\mathbf{w} = \mathbf{r} - \mathbf{p}$
$v = \mathbf{a} \cdot \mathbf{w} = \|\mathbf{a}\|\|\mathbf{w}\| \cos(\theta)$

$$\cos(\theta) = \frac{d}{\|\mathbf{w}\|} \quad \text{then} \quad v = \|\mathbf{a}\|d$$

$$d = \frac{ar_1 + br_2 + c}{\|\mathbf{a}\|}$$

Distance of a Point to a Line



Starting with a Parametric Line

Given: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$ and a point $\mathbf{r}$
Find: $d(\mathbf{r}, \mathbf{l})$
Solution: Let $\mathbf{w} = \mathbf{r} - \mathbf{p}$

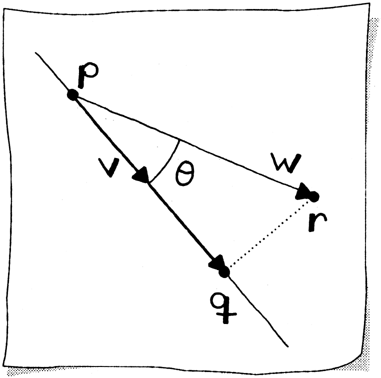
$$d = \|\mathbf{w}\| \sin(\alpha)$$

$$\sin(\alpha) = \sqrt{1 - \cos(\alpha)^2}$$

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

The Foot of a Point

Which point on the line is closest to a given point?
This point is the foot of the given point



Given: $\mathbf{l}(t) = \mathbf{p} + t\mathbf{v}$ and point $\mathbf{r}$
Find: $\mathbf{q}$, the foot of $\mathbf{r}$
Solution: t such that $\mathbf{q} = \mathbf{p} + t\mathbf{v}$
Define $\mathbf{w} = \mathbf{r} - \mathbf{p}$

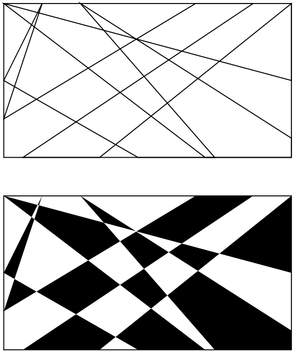
$$\cos(\theta) = \frac{\|t\mathbf{v}\|}{\|\mathbf{w}\|} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

Solve for t

$$t = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2}$$

A Meeting Place: Computing Intersections

Intersection problems important in many applications
Fun figure: finding intersections to create an artistic image

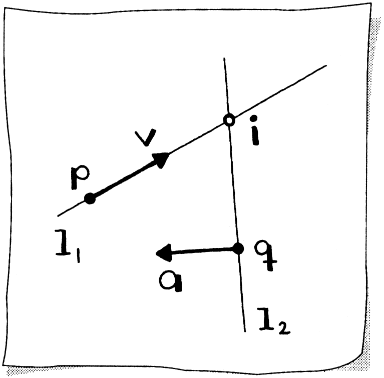


Possible questions:

- Do the lines intersect?
- At what point do they intersect?
- At what parameter value on one or both lines is the intersection point?

Question(s) + line representation(s)
$\Rightarrow$ best method

Parametric and Implicit



Given:

$$\mathbf{l}_1(t) = \mathbf{p} + t\mathbf{v}$$

$$\mathbf{l}_2 : ax_1 + bx_2 + c = 0$$

Find: intersection point $\mathbf{i}$

Solution: $\mathbf{i} = \mathbf{p} + \hat{t}\mathbf{v}$

$a[p_1 + \hat{t}v_1] + b[p_2 + \hat{t}v_2] + c = 0$

One equation and one unknown

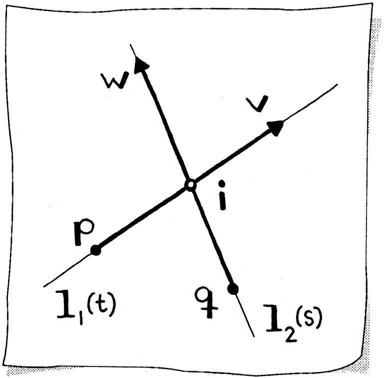
$$\hat{t} = \frac{-c - ap_1 - bp_2}{av_1 + bv_2}$$

$\mathbf{i} = \mathbf{l}_1(\hat{t})$

What if the lines are parallel?

A Meeting Place: Computing Intersections

Both Parametric



Given:

$$\mathbf{l}_1(t) = \mathbf{p} + t\mathbf{v} \qquad \mathbf{l}_2(s) = \mathbf{q} + s\mathbf{w}$$

Find: intersection point $\mathbf{i}$
Solution: $\hat{t}$ and $\hat{s}$ such that

$$\mathbf{p} + \hat{t}\mathbf{v} = \mathbf{q} + \hat{s}\mathbf{w}.$$

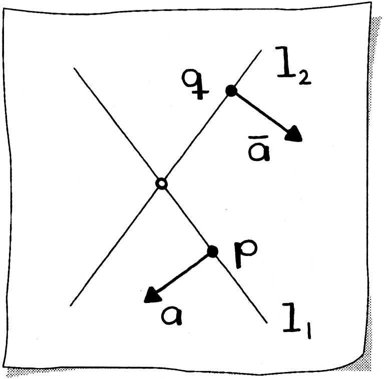
Rewritten:

$$\hat{t}\mathbf{v} - \hat{s}\mathbf{w} = \mathbf{q} - \mathbf{p}$$

Two equations and two unknowns

If $\mathbf{v}$ and $\mathbf{w}$ linearly dependent: no solution
Geometric meaning?

Both Implicit



Given:

$$\mathbf{l}_1 : a x_1 + b x_2 + c = 0$$
$$\mathbf{l}_2 : \bar{a} x_1 + \bar{b} x_2 + \bar{c} = 0$$

Find: intersection point $\mathbf{i} = \hat{\mathbf{x}}$
that satisfies $\mathbf{l}_1$ and $\mathbf{l}_2$
Solution:

$$a\hat{x}_1 + b\hat{x}_2 = -c$$
$$\bar{a}\hat{x}_1 + \bar{b}\hat{x}_2 = -\bar{c}$$

Two equations and two unknowns

Lines parallel
$\Rightarrow$ $\mathbf{a}$ and $\bar{\mathbf{a}}$ linearly dependent

WYSK

- parametric form of a line
- linear interpolation
- point normal form
- implicit form of a line
- explicit form of a line
- line through two points
- line defined by a point and a vector parallel to the line
- line defined by a point and a vector perpendicular to the line
- distance of a point to a line
- line form conversions
- foot of a point
- intersection of lines