Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 6: Moving Things Around: Affine Maps in 2D

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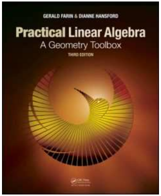
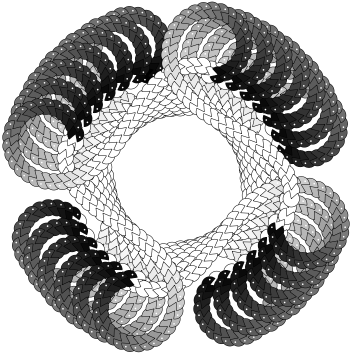
www.farinhansford.com/books/pla



Outline

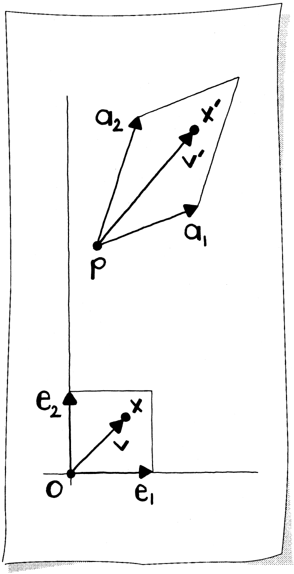
Introduction to Affine Maps in 2D

Moving things around: affine maps in 2D applied to an old and familiar video game character



Imagine playing a video game
Objects are moving around
— shift position, rotate, zoom in/out

These transformations are *affine maps*



Linear maps take $\mathbf{v}$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system to $\mathbf{v}'$ in the $[\mathbf{a}_1, \mathbf{a}_2]$-system:

$$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 = A\mathbf{v}$$

Skew target box defined by a point $\mathbf{p}$ and vectors $\mathbf{a}_1, \mathbf{a}_2$

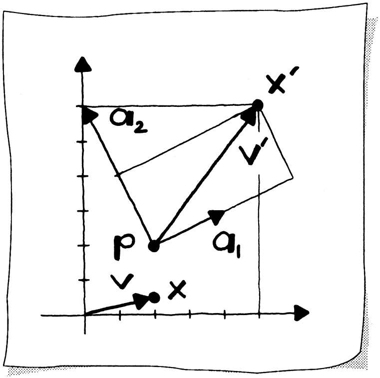
Affine map:

point $\mathbf{x}$ mapped to point $\mathbf{x}'$ by

$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o})$$

Translation + linear map

(Will omit $\mathbf{o}$ when $\mathbf{0}$)



Example: define $[\mathbf{a}_1, \mathbf{a}_2]$-system

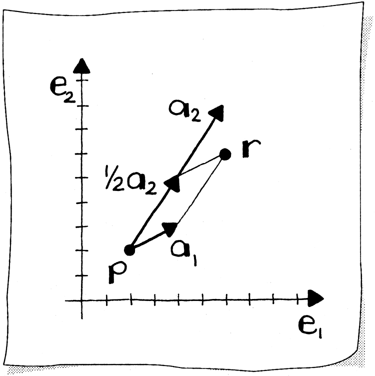
$$\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Point $\mathbf{x} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system

In $[\mathbf{a}_1, \mathbf{a}_2]$-system: coordinates of $\mathbf{x}$ define new point $\mathbf{x}'$

With respect to the $[\mathbf{e}_1, \mathbf{e}_2]$-system:

$$\mathbf{x}' = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$



Example: define $[\mathbf{a}_1, \mathbf{a}_2]$-system

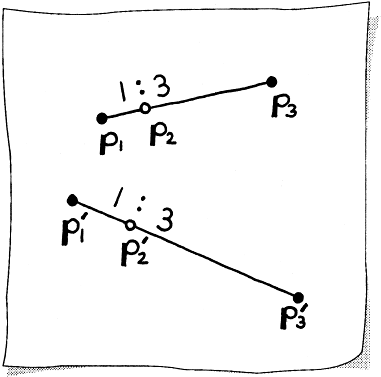
$$\mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Point $\mathbf{r} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ in $[\mathbf{e}_1, \mathbf{e}_2]$-system
What are coordinates of $\mathbf{r}$ in $[\mathbf{a}_1, \mathbf{a}_2]$-system?
Linear system $A\mathbf{u} = (\mathbf{r} - \mathbf{p})$:

$$\begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

Ratios are invariant under linear maps and affine maps



Let $\mathbf{p}_2 = (1-t)\mathbf{p}_1 + t\mathbf{p}_3$

Affine map: $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$

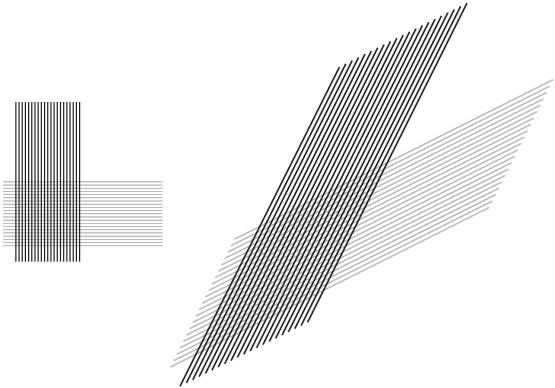
$$\begin{aligned}\mathbf{p}_2' &= A((1-t)\mathbf{p}_1 + t\mathbf{p}_3) + \mathbf{p} \\ &= (1-t)A\mathbf{p}_1 + tA\mathbf{p}_3 \\ &+ [(1-t) + t]\mathbf{p} \\ &= (1-t)[A\mathbf{p}_1 + \mathbf{p}] + t[A\mathbf{p}_3 + \mathbf{p}] \\ &= (1-t)\mathbf{p}_1' + t\mathbf{p}_3'.\end{aligned}$$

Makes use of fact $(1-t) + t = 1$

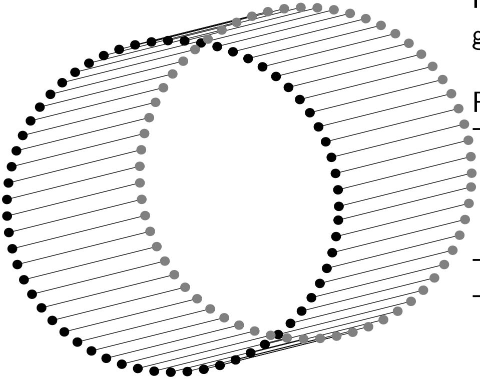
Combining points using *barycentric combinations*

Affine maps map parallel lines to parallel lines

- If two lines do not intersect before mapped, then will not intersect afterwards
- Two lines that intersect before the map will also intersect afterwards



If an object is moved without changing its orientation, then it is translated



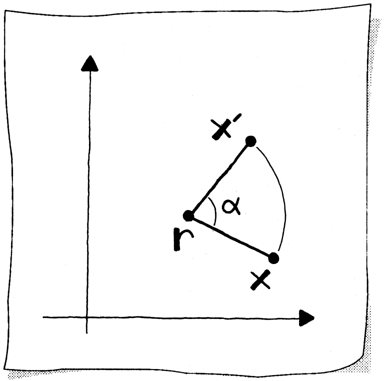
How is this action covered by the general affine map?

Recall *identity matrix* and $I\mathbf{x} = \mathbf{x}$
Translation as a general affine map:

$$\mathbf{x}' = \mathbf{p} + I\mathbf{x}$$

Translations do not change areas
Translation is a *rigid body motion*

Constructive approach to affine maps



Problem: Rotate point $\mathbf{x}$ by α degrees about point $\mathbf{r}$

Translate given geometry:

$$\bar{\mathbf{r}} = \mathbf{r} - \mathbf{r} = \mathbf{0}, \quad \bar{\mathbf{x}} = \mathbf{x} - \mathbf{r}$$

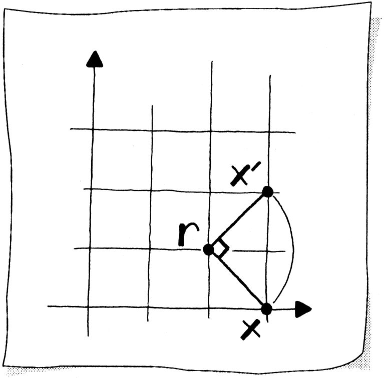
Rotate vector $\bar{\mathbf{x}}$ α degrees:

$$\bar{\bar{\mathbf{x}}} = A\bar{\mathbf{x}}$$

Translate $\bar{\bar{\mathbf{x}}}$ back to the center $\mathbf{r}$ of rotation:

$$\mathbf{x}' = A\bar{\mathbf{x}} + \mathbf{r} = A(\mathbf{x} - \mathbf{r}) + \mathbf{r}$$

Rotate point **x** by α degrees about point **r**



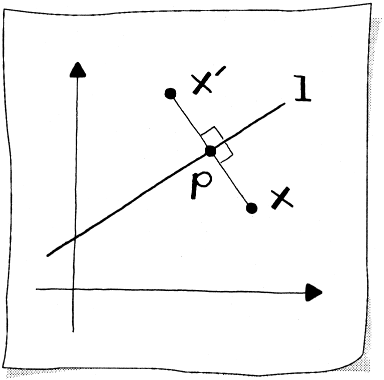
Example:

$$\mathbf{r} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \alpha = 90^\circ$$

Solution:

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Constructive approach to affine maps



Problem: Let **l** be a line and **x** be a point
Reflect **x** across **l** resulting in $\mathbf{x}'$

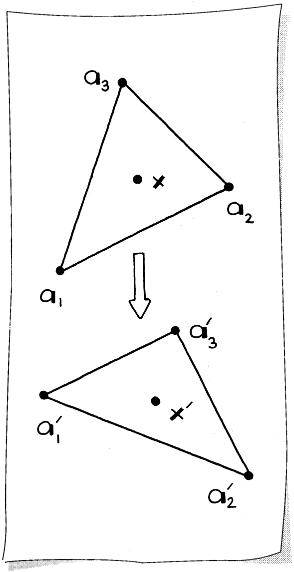
Foot of a point algorithm: find closest point **p** on a line **l** to **x**
p is midpoint of **x** and $\mathbf{x}'$:

$$\mathbf{p} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}'$$

$$\mathbf{x}' = 2\mathbf{p} - \mathbf{x}$$

Best approach not always in standard affine map form

Mapping Triangles to Triangles



Given:

— Triangle T with vertices $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

— Triangle T' with vertices $\mathbf{a}'_1, \mathbf{a}'_2, \mathbf{a}'_3$

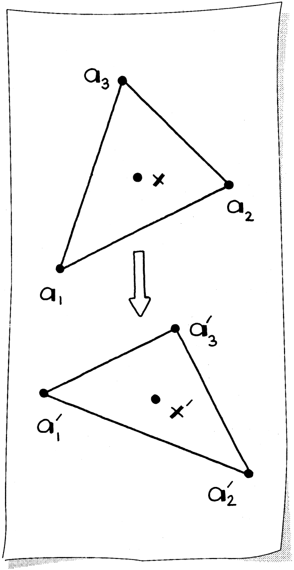
What affine map takes T to T'?

Point $\mathbf{x}$ in T mapped
to which point $\mathbf{x}'$ in T'?

Find A:

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{a}_1] + \mathbf{a}'_1$$

Mapping Triangles to Triangles



Given $\mathbf{x}$ in T, find $\mathbf{x}'$ in T'

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{a}_1] + \mathbf{a}_1'$$

Steps to find A:

$$\mathbf{v}_2 = \mathbf{a}_2 - \mathbf{a}_1 \qquad \mathbf{v}_3 = \mathbf{a}_3 - \mathbf{a}_1$$

$$\mathbf{v}_2' = \mathbf{a}_2' - \mathbf{a}_1' \qquad \mathbf{v}_3' = \mathbf{a}_3' - \mathbf{a}_1'$$

$$A\mathbf{v}_2 = \mathbf{v}_2' \qquad A\mathbf{v}_3 = \mathbf{v}_3'$$

Combine into one matrix equation:

$$A \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_2' & \mathbf{v}_3' \end{bmatrix}$$

$$AV = V'$$

$$A = V'V^{-1}$$

Mapping Triangles to Triangles

Example:

Triangle T: $\mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Triangle T': $\mathbf{a}'_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}'_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \mathbf{a}'_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Construct matrices

$$V = \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad V' = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -1/2 & -1/4 \\ 1/2 & -1/4 \end{bmatrix}$$

$$A = V'V^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Recognize the map?

Mapping Triangles to Triangles

Example continued:

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{a}_1] + \mathbf{a}_1'$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

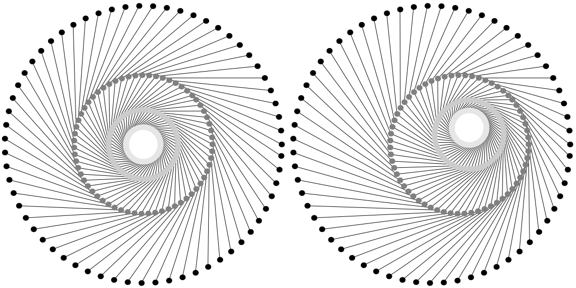
Sample point $\mathbf{x} = \begin{bmatrix} 0 \\ -1/3 \end{bmatrix}$ in T mapped to

$$\mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \left[\begin{bmatrix} 0 \\ -1/3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7/3 \end{bmatrix}$$

in T'

Composing Affine Maps

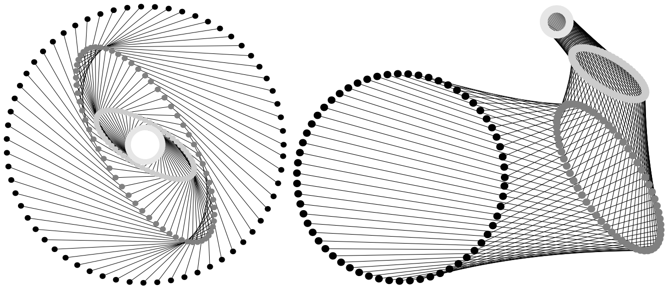
Affine map: $\mathbf{x}' = A(\mathbf{x} - \mathbf{o}) + \mathbf{p}$ Apply twice: $\mathbf{x}'' = A(\mathbf{x}' - \mathbf{o}) + \mathbf{p}$
Repeat several times $\rightarrow$ interesting images



Left:affine map is defined by

$$A = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 0.5 & 0.0 \\ 0 & 0.5 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Right: A and translation $\mathbf{p} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$

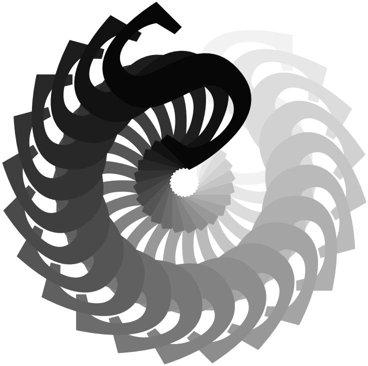


Left: affine map defined by

$$A = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} 0.5 & 0.0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Right: A and translation $\mathbf{p} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Rotations: the letter **S** is rotated several times
— the origin is at the lower left of the letter



Composing Affine Maps

Rotations: the letter **S** is rotated several times

A nonuniform scaling and translation is also applied:

$$\mathbf{x}' = S[R\mathbf{x} + \mathbf{p}]$$

WYSK

- linear map
- affine map
- translation
- identity matrix
- barycentric combination
- invariant ratios
- rigid body motion
- rotate a point about another point
- reflect a point about a line
- three points mapped to three points
- mapping triangles to triangles