Practical Linear Algebra: A Geometry Toolbox

Third edition

Chapter 9: Linear Maps in 3D

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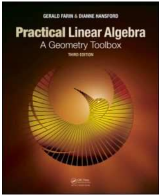


Outline

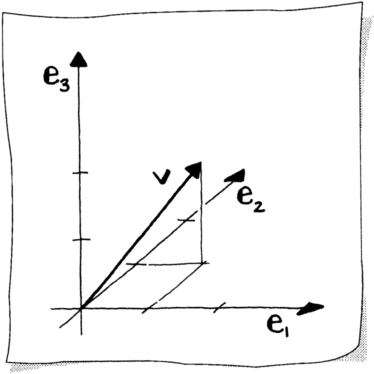
Linear Maps in 3D

Flight simulator: 3D linear maps are necessary to create the twists and turns in a flight simulator(Image is from NASA)

Change the (simulated) position of your plane — simulation software must recompute a new view of the terrain, clouds, or other aircraft

Done through the application of 3D affine and linear maps

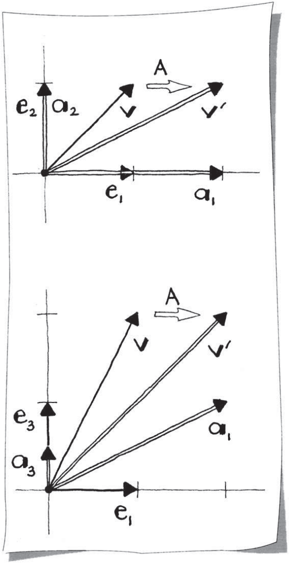
Matrices and Linear Maps



General concept of a linear map in 3D same as that for 2D

Let **v** be a vector in the standard $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-coordinate system

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$



$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$-coordinate system:
origin $\mathbf{0}$ and vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$
What vector $\mathbf{v}'$ in the $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$-system corresponds to $\mathbf{v}$ in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system?

$$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3$$

Example:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{a}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$\mathbf{v}' = 1 \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Matrix equation in 3D: $\mathbf{v}' = A\mathbf{v}$

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

All matrix properties from Linear Maps in 2D (Chapter 4) carry over almost verbatim

Returning to example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Multiply a matrix A by a vector $\mathbf{v}$: the ith component of the result vector obtained as the dot product of the ith row of A and $\mathbf{v}$

Transpose A^{T} of a matrix A
Same idea as 2D: interchange rows and columns

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & -\mathbf{4} \\ 3 & 9 & -4 \\ -1 & -9 & 4 \end{bmatrix}^T = \begin{bmatrix} \mathbf{2} & 3 & -1 \\ \mathbf{3} & 9 & -9 \\ -\mathbf{4} & -4 & 4 \end{bmatrix}$$

Boldface row of A has become the boldface column of A^{T}:

$$a^{\mathrm{T}}_{i,j} = a_{j,i}.$$

Linear Spaces

Set of all 3D vectors is referred to as a
3D *linear space* or *vector space*— denoted as $\mathbb{R}^3$

We associate with $\mathbb{R}^3$ the operation of forming linear combinations
$\Rightarrow$ if **v** and **w** are two vectors in this linear space, then any vector

$$\mathbf{u} = r\mathbf{v} + s\mathbf{w}$$

is also in this space
u is a *linear combination* of **v** and **w**
— combines scalar multiplication and vector addition
This is also called the linearity property

With arbitrary scalars s, t — consider all vectors

$$\mathbf{u} = s\mathbf{v} + t\mathbf{w}$$

They form a subspace of the linear space of all 3D vectors

If vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are in this space then

$$\mathbf{u}_1 = s_1\mathbf{v} + t_1\mathbf{w} \quad \text{and} \quad \mathbf{u}_2 = s_2\mathbf{v} + t_2\mathbf{w}$$

And any linear combination can be written as

$$\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = (\alpha s_1 + \beta s_2)\mathbf{v} + (\alpha t_1 + \beta t_2)\mathbf{w}$$

which is again in the same space

Linear Spaces

Subspace $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ has *dimension* 2 since it is *spanned* by two vectors
These vectors have to be non-collinear
— otherwise they define a line, or a 1D subspace

Example: orthogonal projection of $\mathbf{w}$ onto $\mathbf{v}$ — projection lives in 1D subspace formed by $\mathbf{v}$
If vectors $\mathbf{v}, \mathbf{w}$ collinear — called linearly dependent and $\mathbf{v} = s\mathbf{w}$
If they are not collinear — called linearly independent

Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, then no solution set s_1, s_2 for

$$\mathbf{v}_3 = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$$

Only way to express the zero vector

$$\mathbf{0} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + s_3\mathbf{v}_3 \quad \text{is if} \quad s_1 = s_2 = s_3 = 0$$

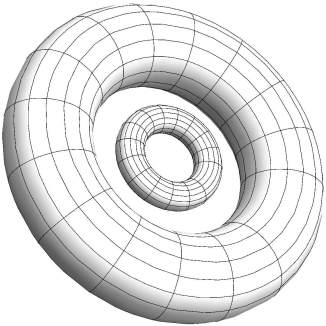
Linear Spaces

Three linearly independent vectors in $\mathbb{R}^3$ span the entire space
The vectors form a basis for $\mathbb{R}^3$

Given linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$
Is $\mathbf{u}$ is in the subspace spanned by $\mathbf{v}$ and $\mathbf{w}$?
— Calculate the volume of the parallelepiped formed by $\mathbf{u}, \mathbf{v}, \mathbf{w}$
— Check if volume is zero (within a round-off tolerance)

Scalings

Scalings in 3D: the large torus is scaled by $1/3$ in each coordinate to form the small torus



Scaling is a linear map which enlarges or reduces vectors:

$$\mathbf{v}' = \begin{bmatrix} s_{1,1} & 0 & 0 \\ 0 & s_{2,2} & 0 \\ 0 & 0 & s_{3,3} \end{bmatrix} \mathbf{v}$$

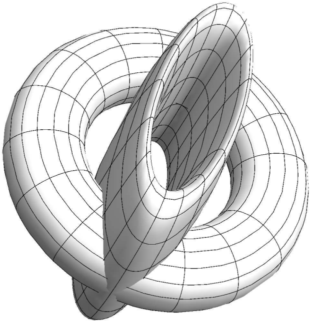
All scale factors

$s_{i,i} > 1$ all vectors enlarged

$0 < s_{i,i} < 1$ all vectors shrunk

Scalings

Non-uniform scalings in 3D: the "standard" torus is scaled by 1/3, 1, 3 in the $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$-directions, respectively



Scaling matrix

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

How do scalings affect volumes?
Unit cube given by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
$\Rightarrow$ Volume 1
Scale
Rectangular box with side lengths
$s_{1,1}, s_{2,2}, s_{3,3}$
$\Rightarrow$ Volume is $s_{1,1}s_{2,2}s_{3,3}$

Scalings

2D: Geometric understanding of the map through illustrations of the *action ellipse*

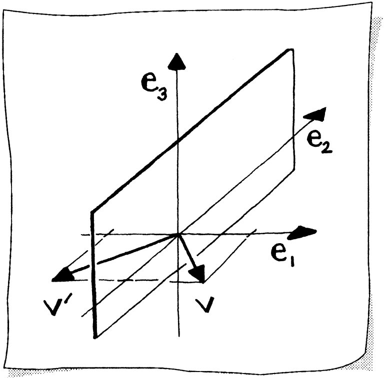
3D: Examine what happens to 3D unit vectors forming a sphere
Mapped to an ellipsoid—the action ellipsoid

- For uniform scale $s_{i,i} = 1/3$: a sphere that is smaller than the unit sphere
- For non-uniform scale $s_{1,1} = 1/3, s_{2,2} = 1, s_{3,3} = 3$: an ellipsoid with major axis in the $\mathbf{e}_3$-direction and minor axis in the $\mathbf{e}_1$-direction

Study the action ellipsoid in more detail in Chapter 16 (The Singular Value Decomposition)

Reflections

Reflection of a vector about the $\mathbf{e}_2, \mathbf{e}_3$-plane



First component should change in sign:

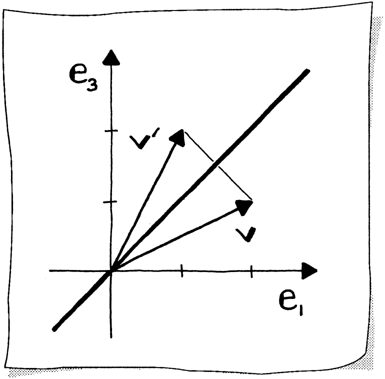
$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \longrightarrow \begin{bmatrix} -v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This reflection achieved by scaling matrix:

$$\begin{bmatrix} -v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Reflections

Reflection of a vector about the $x_1 = x_3$ plane



Interchange the first and third component of a vector

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \longrightarrow \begin{bmatrix} v_3 \\ v_2 \\ v_1 \end{bmatrix}$$

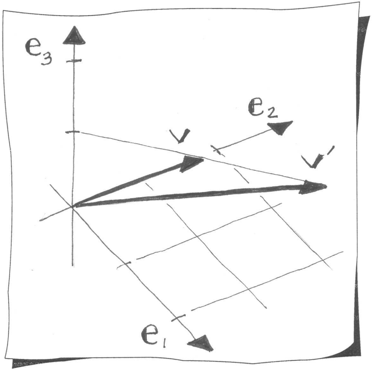
Map achieved by

$$\begin{bmatrix} v_3 \\ v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Reflections do not change volumes —but they do change their signs

Shears

A 3D shear parallel to the $\mathbf{e}_1, \mathbf{e}_2$-plane



A shear maps a cube to a parallelepiped

A shear that maps $\mathbf{e}_1$ and $\mathbf{e}_2$ to themselves and $\mathbf{e}_3$ to $\mathbf{a}_3 = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$

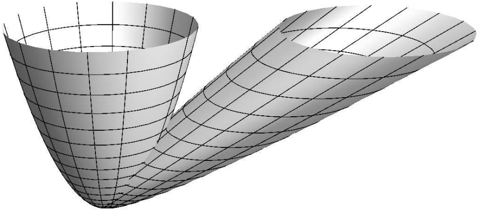
$$S_1 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{v}' = S_1 \mathbf{v} = \begin{bmatrix} v_1 + av_3 \\ v_2 + bv_3 \\ v_3 \end{bmatrix}$$

Sketch: $a = 1, b = 1$

Shears

Shears in 3D: a paraboloid is sheared in the $\mathbf{e}_1$- and $\mathbf{e}_2$-directions
The $\mathbf{e}_3$-direction runs through the center of the left paraboloid



(Same shear as previous sketch)

$$S_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{v}' = S\mathbf{v} = \begin{bmatrix} v_1 + av_3 \\ v_2 + bv_3 \\ v_3 \end{bmatrix}$$

Shears

What shear maps $\mathbf{e}_2$ and $\mathbf{e}_3$ to themselves, and also maps

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ to } \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}?$$

This shear is given by the matrix

$$S_2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-b}{a} & 1 & 0 \\ \frac{-c}{a} & 0 & 1 \end{bmatrix}$$

This map shears parallel to the $[\mathbf{e}_2, \mathbf{e}_3]$-plane
This is the shear of the Gauss elimination step
— See Chapter 12

Shears

Possible to shear in any direction
More common to shear parallel to a coordinate axis or coordinate plane

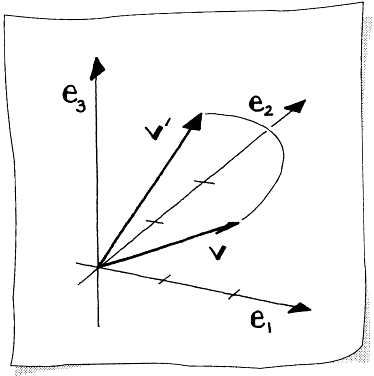
Another example: Shear parallel to the $\mathbf{e}_1$-axis

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + av_2 + bv_3 \\ v_2 \\ v_3 \end{bmatrix}$$

All shears are volume preserving

Rotations

Rotate a vector $\mathbf{v}$ around the $\mathbf{e}_3$-axis by 90° to a vector $\mathbf{v}'$:



$$\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \mathbf{v}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

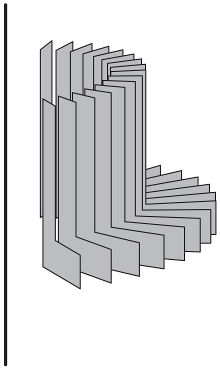
Rotation around $\mathbf{e}_3$ by any angle leaves third component unchanged

Desired rotation matrix R_3:
(similar to one from 2D)

$$R_3 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations

Rotations in 3D: the letter "L" rotated about the $\mathbf{e}_3$-axis



Verify that R_3 performs as promised with $\alpha = 90^\circ$:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Rotations

Rotate around the $\mathbf{e}_2$-axis:

$$R_2 = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Notice the pattern: Rotation about $\mathbf{e}_i$-axis
$\Rightarrow$ i^{th} row is $\mathbf{e}_i$ and i^{th} column is $\mathbf{e}_i^{\mathrm{T}}$

Rotation around the $\mathbf{e}_1$-axis:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

Positive angle rotation follows the right-hand rule:
curl your fingers with the rotation, and your thumb points in the direction of the rotation axis

Rotations

Example: Rotation matrix about the $\mathbf{e}_1$-axis:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

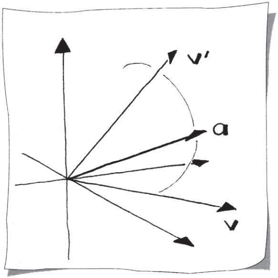
Column vectors form an *orthonormal* set of vectors
— Each column vector is a unit length vector
— They are orthogonal to each other
$\Rightarrow$ A rotation matrix is an *orthogonal matrix*
(These properties hold for the row vectors of the matrix too.)

$$R^{\mathrm{T}}R = I \qquad R^{\mathrm{T}} = R^{-1}$$

If R rotates by θ then R^{-1} rotates by $-\theta$
Rotations do not change volumes
Rotations are *rigid body motions*

Rotations

Rotation α degrees about an arbitrary vector **a**



$$R = \begin{bmatrix} a_1^2 + C(1-a_1^2) & a_1a_2(1-C) - a_3S & a_1a_3(1-C) + a_2S \\ a_1a_2(1-C) + a_3S & a_2^2 + C(1-a_2^2) & a_2a_3(1-C) - a_1S \\ a_1a_3(1-C) - a_2S & a_2a_3(1-C) + a_1S & a_3^2 + C(1-a_3^2) \end{bmatrix}$$

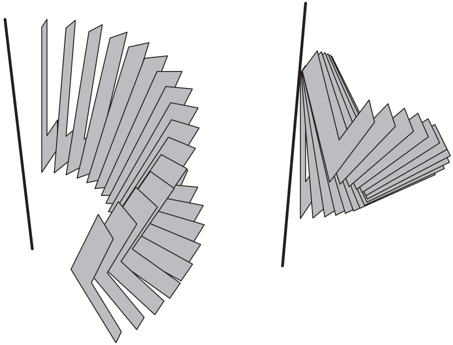
$C = \cos\alpha$ and $S = \sin\alpha$

Necessary that $\|\mathbf{a}\| = 1$ to avoid scaling

(Derivation is a bit tricky!)

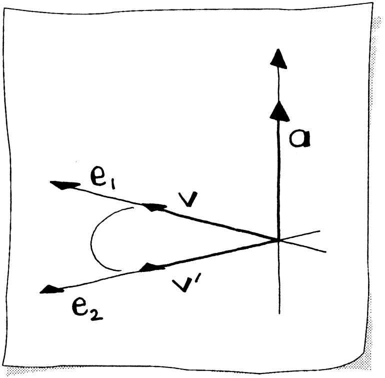
Rotations

Rotations in 3D: the letter "L" is rotated about axes that are not the coordinate axes
Right: the point on the "L" that touches the rotation axes does not move



Rotations

A simple example of a rotation about a vector



$$\alpha = 90^\circ \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(In advance — we know R)
$C = 0$ and $S = 1$ then

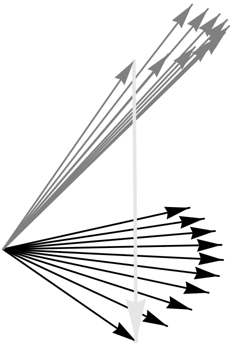
$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

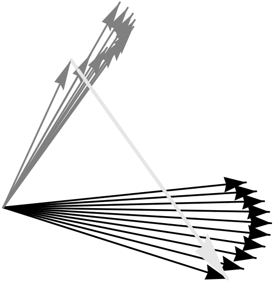
$$\mathbf{v}' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Parallel projections in 3D

Left: orthogonal projection   Right: oblique projection of 45°

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$





Projections

Parallel projections are linear maps
(Perspective projection are *not* linear maps)

Parallel projections preserve relative dimensions of an object
$\Rightarrow$ used in drafting to produce accurate views of a design

Recall from 2D: a projection matrix P
— Reduces dimensionality (flattens) because P is rank deficient
In 3D: a vector is projected into a subspace $\Rightarrow$ (2D) plane or (1D) line
— Is an idempotent map $P\mathbf{v} = P^2\mathbf{v}$
Leaves a vector in the subspace of the map unchanged by the map

Projections

Construction of an orthogonal projection in 3D

Choose the subspace U into which to project
— Line: specify a unit vector $\mathbf{u}_1$
— Plane: specify two orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2$

Form matrix A_k from the vectors defining the k-dimensional subspace U:

$$A_1 = \mathbf{u}_1 \quad \text{or} \quad A_2 = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

Projection matrix P_k:

$$P_k = A_k A_k^{\mathrm{T}}$$

P_1 very similar to the projection matrix from 2D:

$$P_1 = A_1 A_1^{\mathrm{T}} = \begin{bmatrix} u_{1,1}\mathbf{u}_1 & u_{2,1}\mathbf{u}_1 & u_{3,1}\mathbf{u}_1 \end{bmatrix}$$

Projections

Projection into a plane:

$$P_2 = A_2 A_2^{\mathrm{T}} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \mathbf{u}_2^{\mathrm{T}} \end{bmatrix}$$

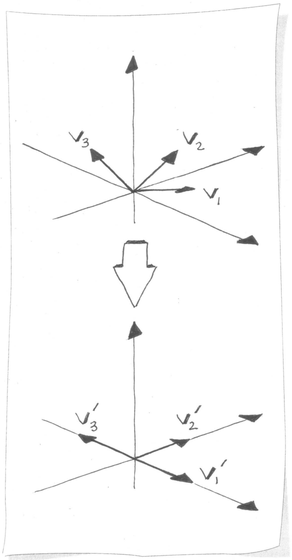
Expanding — columns of P_2 are linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$

$$P_2 = \begin{bmatrix} u_{1,1}\mathbf{u}_1 + u_{1,2}\mathbf{u}_2 & u_{2,1}\mathbf{u}_1 + u_{2,2}\mathbf{u}_2 & u_{3,1}\mathbf{u}_1 + u_{3,2}\mathbf{u}_2 \end{bmatrix}$$

The action of P_1 and P_2:

$$P_1\mathbf{v} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \qquad P_2\mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2$$

Projections



Construct orthogonal projection P_2 into the $[\mathbf{e}_1, \mathbf{e}_2]$-plane

$$P_2 = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}\begin{bmatrix}\mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

Action of the map:

$$\begin{bmatrix}v_1 \\ v_2 \\ 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2 \\ v_3\end{bmatrix}$$

Projection direction is $\mathbf{d} = [0\ 0\ \pm 1]^{\mathrm{T}}$
$P_2\mathbf{d} = \mathbf{0} \Rightarrow$ projection direction is in the kernel of the map

Projections

The idempotent property for P_2:

$$\begin{aligned} P_2^2 &= A_2 A_2^{\mathrm{T}} A_2 A_2^{\mathrm{T}} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \mathbf{u}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \mathbf{u}_2^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} I \begin{bmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \mathbf{u}_2^{\mathrm{T}} \end{bmatrix} \\ &= P_2 \end{aligned}$$

Orthogonal projection matrices are symmetric:

Action of the map $P\mathbf{v}$ is orthogonal to $\mathbf{v} - P\mathbf{v}$

$$0 = (P\mathbf{v})^{\mathrm{T}}(\mathbf{v} - P\mathbf{v}) = \mathbf{v}^{\mathrm{T}}(P^{\mathrm{T}} - P^{\mathrm{T}}P)\mathbf{v} \quad \rightarrow P = P^{\mathrm{T}}$$

Projection results in zero volume

Volumes and Linear Maps: Determinants

Volume change is an important aspect of the action of a map

Unit cube in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system has volume one
Linear map A will change cube to a skew box spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$
—the column vectors of A
What is the volume spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$?

Recall 2×2 matrix
 Area of a 2D parallelogram equivalent to a determinant
 Cross product can be used to calculate this area (by embedding in 3D)

3D geometry: *scalar triple product*
 Calculate volume of a parallelepiped using a "base area times height"

Revisit this and look at it from the perspective of linear maps

Volumes and Linear Maps: Determinants

3×3 determinant of a matrix A — alternating sum of 2×2 determinants:

$$|A| = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{2,1} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} + a_{3,1} \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}$$

Called the cofactor expansion or expansion by minors
— Each (signed) 2×2 determinant is *cofactor* of $a_{i,j}$ it is paired with
— Sign comes from the factor $(-1)^{i+j}$
— Cofactor is also written as $(-1)^{i+j} M_{i,j}$
where $M_{i,j}$ is called the minor of $a_{i,j}$

Volumes and Linear Maps: Determinants

Trick to remember determinant expression:

Copy the first two columns after the last column
Form the product of the three "diagonals" and add them

$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \Box & \Box \\ \Box & a_{2,2} & a_{2,3} & a_{2,1} & \Box \\ \Box & \Box & a_{3,3} & a_{3,1} & a_{3,2} \end{array}$$

Form the product of the three "anti-diagonals" and subtract them

$$\begin{array}{ccccc} \Box & \Box & a_{1,3} & a_{1,1} & a_{1,2} \\ \Box & a_{2,2} & a_{2,3} & a_{2,1} & \Box \\ a_{3,1} & a_{3,2} & a_{3,3} & \Box & \Box \end{array}$$

The complete formula:

$$\begin{aligned} |A| &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ &\quad - a_{3,1}a_{2,2}a_{1,3} - a_{3,2}a_{2,3}a_{1,1} - a_{3,3}a_{2,1}a_{1,2} \end{aligned}$$

What is the volume spanned by the three vectors

$$\mathbf{a}_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.1 \end{bmatrix}?$$

$$\begin{aligned} \det[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] &= 4 \begin{vmatrix} 4 & -0.1 \\ 4 & 0.1 \end{vmatrix} \\ &= 4(4 \times 0.1 - (-0.1) \times 4) = 3.2 \end{aligned}$$

(Did not write down zero terms)
Notice: $\det A$ is alternative notation for $|A|$

Volumes and Linear Maps: Determinants

3D shear preserves volume
Apply series of shears to A resulting in

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \tilde{a}_{1,3} \\ 0 & \tilde{a}_{2,2} & \tilde{a}_{2,3} \\ 0 & 0 & \tilde{a}_{3,3} \end{bmatrix}$$

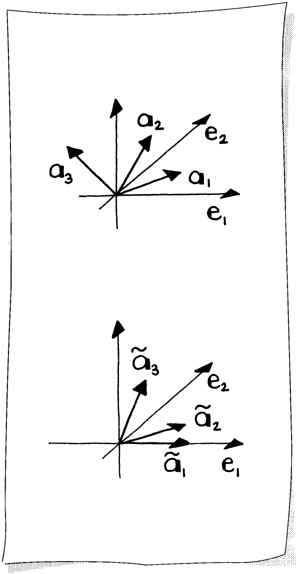
$$|\tilde{A}| = \tilde{a}_{1,1}\tilde{a}_{2,2}\tilde{a}_{3,3} \qquad \text{and} \qquad |A| = |\tilde{A}|$$

Revisit Example above

One simple row operation: $\text{row}_3 = \text{row}_3 - \text{row}_2$ results in

$$\tilde{A} = \begin{bmatrix} 4 & -1 & 0.1 \\ 0 & 4 & -0.1 \\ 0 & 0 & 0.2 \end{bmatrix} \qquad |\tilde{A}| = |A| = 3.2$$

Volumes and Linear Maps: Determinants



Shear/forward elimination concept provides an easy to visualize interpretation of the 3×3 determinant

First two column vectors of $\tilde{A}$ lie in the $[\mathbf{e}_1, \mathbf{e}_2]$-plane

Their determinant defines the area of the parallelogram that they span

— this determinant is $\tilde{a}_{1,1}\tilde{a}_{2,2}$

The height of the skew box is the $\mathbf{e}_3$ component of $\tilde{\mathbf{a}}_3$

This is equivalent to the scalar triple product

Volumes and Linear Maps: Determinants

Rules for determinants $\qquad$ A and B are 3×3 matrices

- $|A| = |A^{\mathrm{T}}| \Rightarrow$ row and column equivalence
- Non-cyclic permutation changes the sign: $\begin{vmatrix}\mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3\end{vmatrix} = -|A|$
- scalar c: $\begin{vmatrix}c\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{vmatrix} = c|A|$
- $|cA| = c^3|A|$
- If A has a row of zeroes then $|A| = 0$
- If A has two identical rows then $|A| = 0$
- $|A| + |B| \neq |A + B|$, in general
- $|AB| = |A||B|$
- Multiples of rows can be added together without changing the determinant. Example: shears of Gauss elimination
- A being invertible is equivalent to $|A| \neq 0$
- If A is invertible then $|A^{-1}| = \frac{1}{|A|}$

Apply a linear map A to a vector $\mathbf{v}$ then apply a map B to the result:

$$\mathbf{v}' = BA\mathbf{v}$$

Matrix multiplication is defined just as in the 2D case

$C = BA$: element $c_{i,j}$ is dot product of B's ith row and A's jth column

	A
B	C

			1	**5**	−4
			−1	**−2**	0
			2	**3**	−4
0	**0**	**−1**	−2	**−3**	4
1	−2	0	3	9	−4
−2	1	1	−1	−9	4

Multiply two matrices A and B together as AB — sizes of A and B:

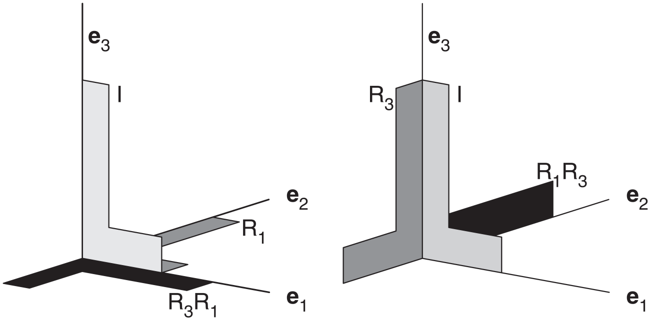
$$m \times n \quad \text{and} \quad n \times p$$

Resulting matrix: $m \times p$—the "outside" dimensions

Necessary that "inside" dimensions (both n) be equal

Combining Linear Maps

Matrix multiplication does not commute (in general): $AB \neq BA$
In 2D rotation commute — In 3D they do not



The original "L" is labeled I for identity matrix
Left: R_1 is applied and then R_3 — result labeled R_3R_1
Right: R_3 is applied and then R_1 — result labeled R_1R_3

Combining Linear Maps

Matrices for last Figure:
R_1: rotation by -90° around the $\mathbf{e}_1$-axis
R_3: rotation by -90° around the $\mathbf{e}_3$-axis

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad R_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 R_1 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{is not equal to} \quad R_1 R_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Inverse Matrices

Inverse matrices undo linear maps:

$$\mathbf{v}' = A\mathbf{v} \qquad \text{then} \quad A^{-1}\mathbf{v}' = \mathbf{v} \quad \text{or} \quad A^{-1}A\mathbf{v} = \mathbf{v}$$

Combined action of A^{-1} and A has no effect on any vector $\mathbf{v}$

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix is not always invertible
Example: projections — they are rank deficient

Inverse Matrices

Orthogonal matrices: constructed from a set of orthonormal vectors
$R^{\mathrm{T}} = R^{-1}$
Forming the reverse rotation is simple and requires no computation
— Provides for huge savings in computer graphics where rotating objects is common

Scaling also has a simple to compute inverse:

$$S = \begin{bmatrix} s_{1,1} & 0 & 0 \\ 0 & s_{2,2} & 0 \\ 0 & 0 & s_{3,3} \end{bmatrix} \quad \text{then} \quad S^{-1} = \begin{bmatrix} 1/s_{1,1} & 0 & 0 \\ 0 & 1/s_{2,2} & 0 \\ 0 & 0 & 1/s_{3,3} \end{bmatrix}$$

Inverse Matrices

Rules calculating with inverse matrices

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1} \cdot \; \ldots \; \cdots A^{-1}}_{n \text{ times}}$$

$$(A^{-1})^{-1} = A$$

$$(kA)^{-1} = \frac{1}{k} A^{-1}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Chapter 12: details on calculating A^{-1}

More on Matrices

Restate some matrix properties — hold for $n \times n$ matrices as well

- preserve scalings: $A(c\mathbf{v}) = cA\mathbf{v}$
- preserve summations: $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- preserve linear combinations: $A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}$
- distributive law: $A\mathbf{v} + B\mathbf{v} = (A + B)\mathbf{v}$

More on Matrices

- commutative law for addition: $A + B = B + A$
- no commutative law for multiplication: $AB \neq BA$
- associative law for addition: $A + (B + C) = (A + B) + C$
- associative law for multiplication: $A(BC) = (AB)C$
- distributive law: $A(B + C) = AB + AC$
  $(B + C)A = BA + CA$

Scalar laws:

- $a(B + C) = aB + aC$
- $(a + b)C = aC + bC$
- $(ab)C = a(bC)$
- $a(BC) = (aB)C = B(aC)$

More on Matrices

Laws involving determinants:

- $|A| = |A^{\mathrm{T}}|$
- $|AB| = |A| \cdot |B|$
- $|A| + |B| \neq |A + B|$
- $|cA| = c^n |A|$

More on Matrices

Laws involving exponents:

- $A^r = \underbrace{A \cdot \ldots \cdots A}_{r \text{ times}}$
- $A^{r+s} = A^r A^s$
- $A^{rs} = (A^r)^s$
- $A^0 = I$

More on Matrices

Laws involving the transpose:

- $[A + B]^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$
- $A^{\mathrm{T}^{\mathrm{T}}} = A$
- $[cA]^{\mathrm{T}} = cA^{\mathrm{T}}$
- $[AB]^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$

WYSK

- 3D linear map
- transpose matrix
- linear space
- vector space
- subspace
- linearity property
- linear combination
- linearly independent
- linearly dependent
- scale
- action ellipsoid
- rotation
- rigid body motions
- shear
- reflection
- projection
- idempotent
- orthographic projection
- oblique projection
- determinant
- volume
- scalar triple product
- cofactor expansion
- expansion by minors
- inverse matrix
- multiply matrices
- non-commutative property of matrix multiplication
- rules of matrix arithmetic