Multivariate Functions

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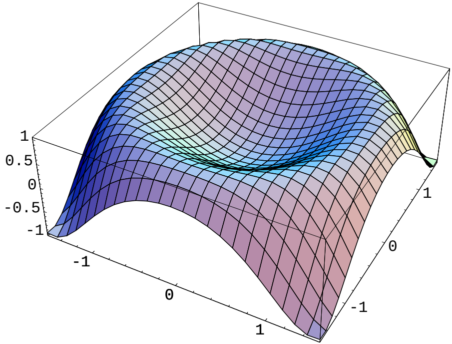
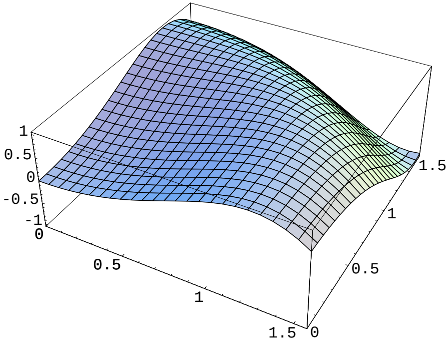
Example

$$z = \sin(x^2 + y^2)$$
$$-\pi/2 \le x \le \pi/2$$
$$-\pi/2 \le y \le \pi/2.$$



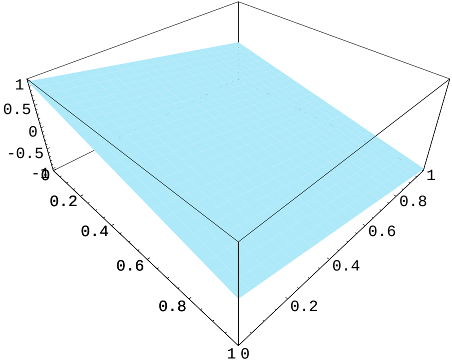
Example

$z = \sin(x^2 + y^2)$
$0 \leq x \leq \pi/2$
$0 \leq y \leq \pi/2$



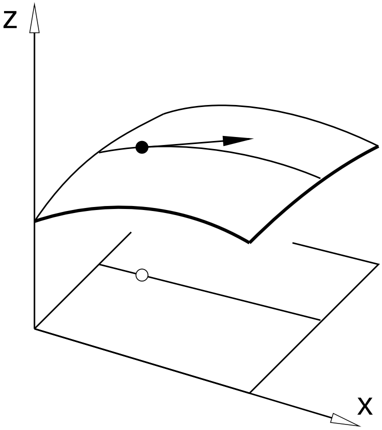
Example: Linear

$$l(x, y) = ax + by + c$$



Partials

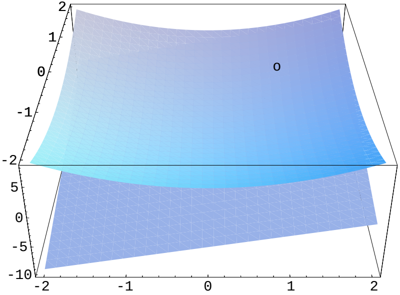
$$\frac{\partial f(x,c)}{\partial x} \qquad \text{or} \qquad f_x(x,c)$$



Tangent Planes

tangent plane at (x_0, y_0):

$$
\begin{aligned}
l(x, y) = \\
f(x_0, y_0) \\
+(x - x_0)f_x(x_0, y_0) \\
+(y - y_0)f_y(x_0, y_0)
\end{aligned}
$$



Gradients

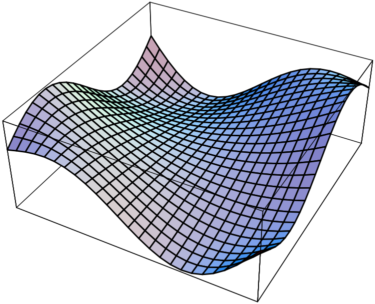
$$\nabla f = (f_x, f_y)$$

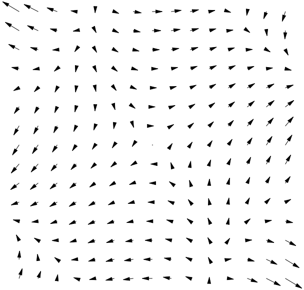
tangent plane:

$$l(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

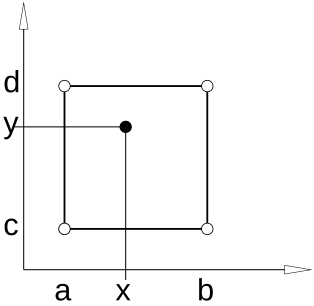
∇f: direction of steepest slope

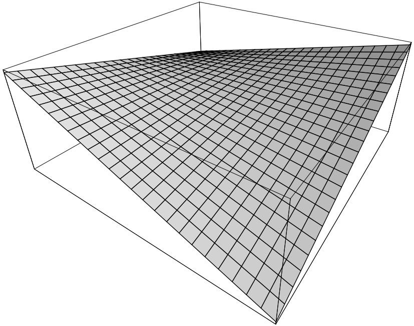
Gradient Field





Bilinear Interpolation





Bilinear Interpolation

$$z = \begin{bmatrix} \frac{b-x}{b-a} & \frac{x-a}{b-a} \end{bmatrix} \begin{bmatrix} z_{a,c} & z_{a,d} \\ z_{b,c} & z_{b,d} \end{bmatrix} \begin{bmatrix} \frac{d-y}{d-c} \\ \frac{y-c}{d-c} \end{bmatrix}$$

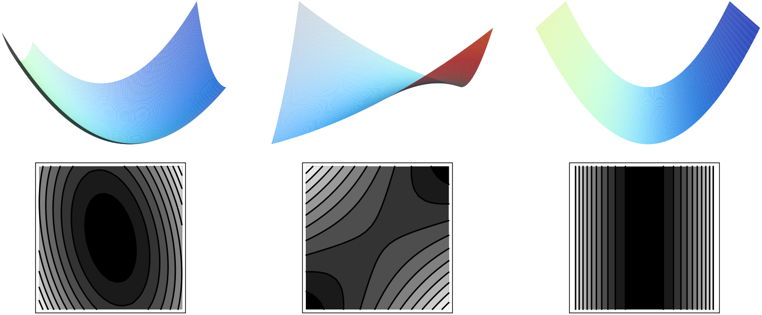
Quadratic Forms

$$q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x} + c$$

A: symmetric 2×2

$$q(\mathbf{x}) = a_{1,1}x^2 + 2a_{1,2}xy + a_{2,2}y^2 + c$$

Quadratic Forms



$$A_1 = \begin{bmatrix} 3 & 0.5 \\ 0.5 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues

A_1: $3.1, 0.89$, $\quad A_2$: $3, -1$, $\quad A_3$: $2, 0$

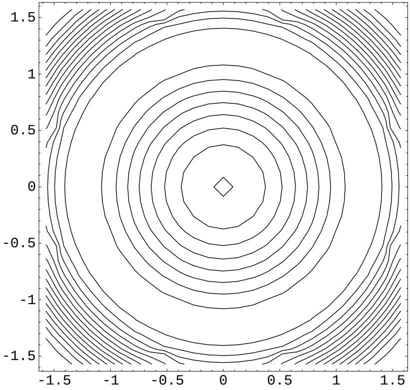
- A_1: positive definite $\rightarrow$ elliptic paraboloid
- A_2: negative definite $\rightarrow$ hyperbolic paraboloid
- A_3: indefinite $\rightarrow$ parabolic cylinder

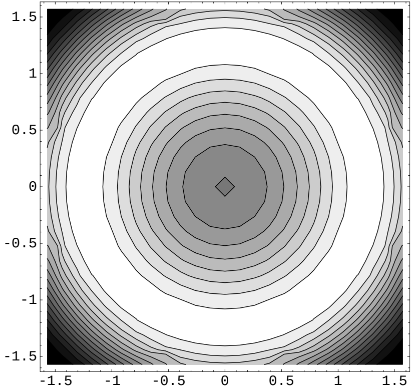
Contours

$$f(x, y) = c$$

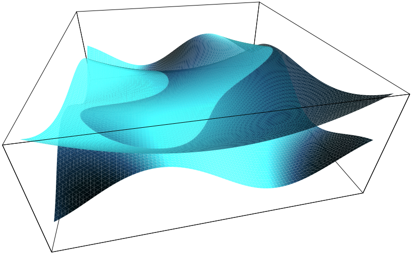
family of 2D curves

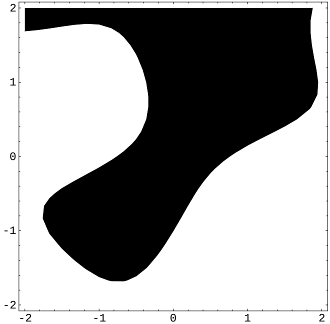
Examples





Example

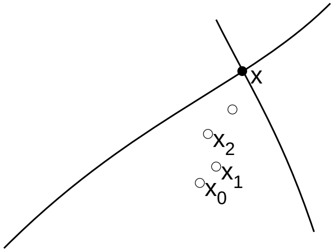




Newton-Raphson

$$\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}$$



Newton-Raphson

- at $\mathbf{x}_0$, replace f_1 by its tangent plane T_1
- at $\mathbf{x}_0$, replace f_2 by its tangent plane T_2
- T_1 intersects x, y-plane in line l_1
- T_2 intersects x, y-plane in line l_2
- $\mathbf{x}_1 =$ intersection of l_1, l_2
- repeat

Newton-Raphson

$$0 = l_1(\mathbf{x}) = f_1(\mathbf{x}_0) + \nabla f_1(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

$$0 = l_2(\mathbf{x}) = f_2(\mathbf{x}_0) + \nabla f_2(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

combine:

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_0) + \begin{bmatrix} \nabla f_1(\mathbf{x}_0) \\ \nabla f_2(\mathbf{x}_0) \end{bmatrix} (\mathbf{x}_1 - \mathbf{x}_0)$$

Jacobian

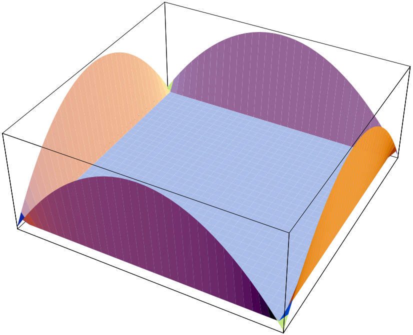
$$J(\mathbf{x}_0) = \begin{bmatrix} \nabla f_1(\mathbf{x}_0) \\ \nabla f_2(\mathbf{x}_0) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x}_0)}{\partial x} & \frac{\partial f_1(\mathbf{x}_0)}{\partial y} \\ \frac{\partial f_2(\mathbf{x}_0)}{\partial x} & \frac{\partial f_2(\mathbf{x}_0)}{\partial y} \end{bmatrix}$$

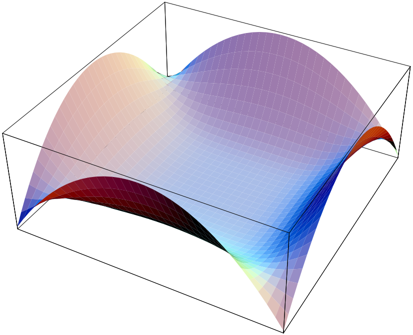
Jacobian

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_0) + J(\mathbf{x}_0) \cdot (\mathbf{x}_1 - \mathbf{x}_0)$$

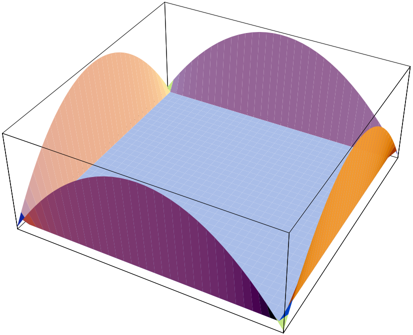
$$\mathbf{x}_1 = \mathbf{x}_0 - J^{-1}(\mathbf{x}_0) \cdot \mathbf{f}(\mathbf{x}_0)$$

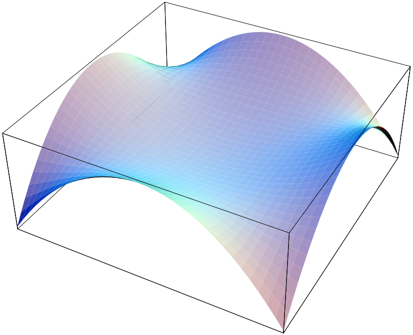
PDE Example





PDE Example





PDE Example

Input: four "wires"

$$f(0, y), \quad f(x, 0), \quad f(1, y), \quad f(x, 1)$$

Output: "soap bubble" $f(x, y)$

PDE:

$$f_{xx}(x, y) + f_{yy}(x, y) = 0$$

Discretization

$$f_{xx}(x_i, y_j) \approx \frac{1}{h^2}[f_{i+1,j} - 2f_{i,j} + f_{i-1,j}],$$
$$f_{yy}(x_i, y_j) \approx \frac{1}{h^2}[f_{i,j+1} - 2f_{i,j} + f_{i,j-1}]$$

Discretization

$$f_{i+1,j} - 2f_{i,j} + f_{i-1,j} + f_{i,j+1} - 2f_{i,j} + f_{i,j-1} = 0$$

$$f_{i,j} = [f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1}]/4$$

Mask:

$$f_{i,j} \quad = \quad \begin{array}{ccc} 0 & 0.25 & 0 \\ 0.25 & \bullet & 0.25 \\ 0 & 0.25 & 0 \end{array}$$

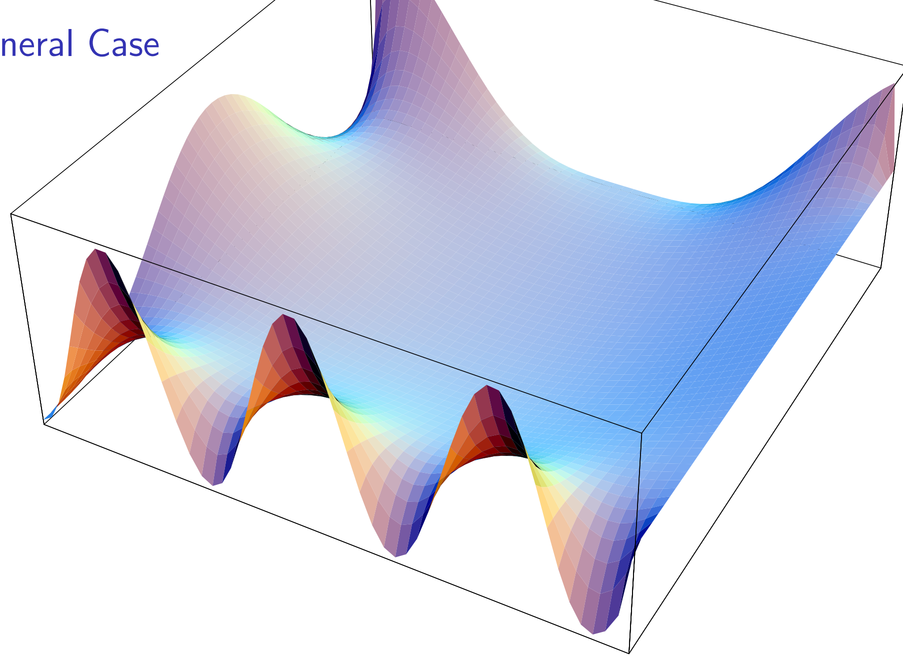
Solution

$$f_{i,j} \quad = \quad \begin{array}{ccc} 0 & 0.25 & 0 \\ 0.25 & \bullet & 0.25 \\ 0 & 0.25 & 0 \end{array}$$

- one linear equation per unknown $f_{i,j}$
- Given values: $f_{0,j}, f_{i,N} \ldots$
- Start: $f_{i,j} = 0$
- iterate

$\Rightarrow$ Iterative solution of a linear system (Gauss-Seidel)

General Case

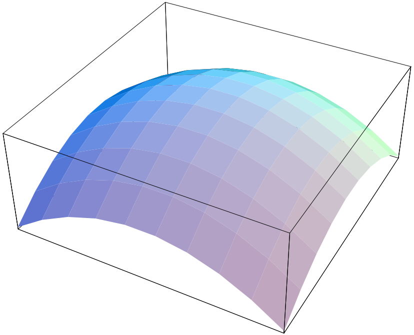


Example: Euler-Lagrange

$$f_{xxyy} = 0$$

Mask:

$$f_{i,j} \quad = \quad \begin{array}{ccc} -0.25 & 0.50 & -0.25 \\ 0.50 & \bullet & 0.50 \\ -0.25 & 0.50 & -0.25 \end{array}$$



Trivariate Functions

$$w = f(x, y, z)$$

$$w = \sin 8x \sin 8y \sin 8z$$

w=0.45