LA Background

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Matrices

- digital images: about 1000×1000
- the Google matrix: about 10 billion by 10 billion
- transformation matrices: 3×3

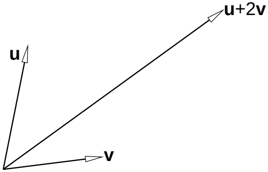
Vectors

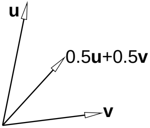
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -2.1 \\ 3 \end{bmatrix}$$

linear combinations:

$$\mathbf{w} = s\mathbf{u} + t\mathbf{v}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} s \cdot u_1 + t \cdot v_1 \\ s \cdot u_2 + t \cdot v_2 \end{bmatrix}.$$





Linear Spaces

- A set for which linear combinations are defined
- and which is closed under those linear combinations

Linear Spaces

real numbers / addition
2×2 matrices / matrix addition

Not Linear Spaces

positive reals / addition
2×2 rotation matrices / matrix addition

Dimensions

General vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Space: $\mathbb{R}^n$.
Dimension: n.

Linear Dependence

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

$$\mathbf{w} = 2\mathbf{u} - \mathbf{v}.$$

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: linearly dependent.

Linear Independence

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\mathbf{w} = \textit{something } \mathbf{u} \;+\; \textit{something } \mathbf{v}? \quad \Rightarrow \quad \text{No}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: linearly independent.

Dimensions

Linear space has dimension n: contains at most n linearly independent vectors.

- 3D vectors: dimension 3
- 2×2 matrices: dimension 4
- the reals: dimension 1

Bases

$\mathbf{u}_1, \ldots, \mathbf{u}_n$ l.i. in $n-$dim. space U^n: **basis**.
$\mathbf{u}_1, \ldots, \mathbf{u}_n$ **span** U^n.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: basis for $\mathbb{R}^2$
- 3: basis for $\mathbb{R}$

Subspaces

$U^m \subset U^n$: U^m is **subspace** of U^n.

- $\mathbb{R}^2$ is subspace of $\mathbb{R}^3$
- $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ span subspace of $\mathbb{R}^3$.

Matrices

$$B = \begin{bmatrix} b_{1,1} & \cdots\cdots & b_{1,m} \\ \vdots & & \vdots \\ b_{n,1} & \cdots\cdots & b_{n,m} \end{bmatrix}$$

$$B = \begin{bmatrix} 1, 2, -1, 3 \\ -1, 4, 0, 0 \\ 0, -2, 2, 1 \end{bmatrix}$$

row:

$$\mathbf{b}_2^{\mathrm{T}} = \begin{bmatrix} -1, 4, 0, 0 \end{bmatrix}$$

column:

$$\mathbf{b}_4 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

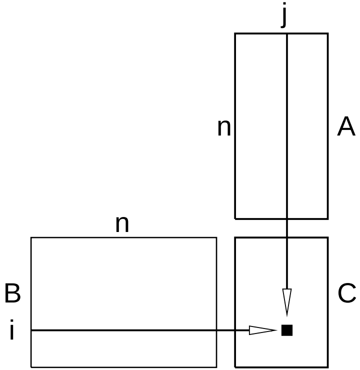
Multiplies I

$$A \cdot \mathbf{u} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{1,1}u_1 + a_{1,2}u_2 \\ a_{2,1}u_1 + a_{2,2}u_2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Multiplies II

$$\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & -3\\ -1 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 4 & -3\end{bmatrix}$$



Linear Maps

A: n rows, m columns.

A defines map: $\mathbb{R}^m \rightarrow \mathbb{R}^n$

A is linear: $A(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha A\mathbf{u} + \beta A\mathbf{v}$.

Dyadic Sums

$$BA = \mathbf{b}_1\mathbf{a}_1^{\mathrm{T}} + \ldots + \mathbf{b}_n\mathbf{a}_n^{\mathrm{T}}. \tag{1}$$

$$\begin{aligned}
\begin{bmatrix} 2 & 1 \\ 4 & -3 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & -6 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

Rank

$$B = [\mathbf{b}_1, \ldots, \mathbf{b}_n].$$

$$\mathrm{rank}(B) = \mathrm{dimension}(\mathbf{b}_1, \ldots, \mathbf{b}_n).$$

$$\mathrm{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = 2, \quad \mathrm{rank} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = 3$$

Inverse

A: square, full rank. **Inverse** A^{-1}:

$$AA^{-1} = I$$

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$$

Length

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \ldots + u_n^2}.$$

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}}.$$

$\mathbf{u}^{\mathrm{T}}\mathbf{u}$: **dot product**
$\mathbf{u}^{\mathrm{T}}\mathbf{u} = 1$: **unit vector**

Dot Products

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathrm{T}}\mathbf{v}.$$

$\mathbf{uw} = 0 : \quad \mathbf{u}, \mathbf{v}$ orthogonal

$$\cos(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Orthogonal Matrices

$U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ square, $\mathbf{u}_i$ unit vectors
$\mathbf{u}_i \mathbf{u}_j = 0$ for $i \neq j$: U is **orthogonal**

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$