Eigen-Problems

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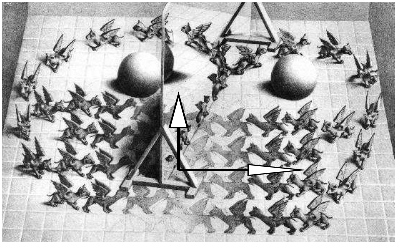
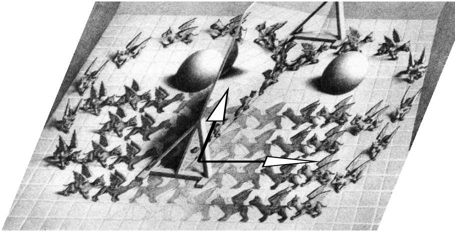
Eigen-Collapse

Eigenvectors





Eigen-Example I

Find $\lambda, \mathbf{u}$ in

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{u} = \lambda \mathbf{u}$$

Mathematica: $\lambda_1 = 3, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 1, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Combine:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Example II

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mathematica:

$$\lambda_1 = i, \lambda_2 = -i, \lambda_3 = 1$$

Rotation matrix: complex eigenvalues.
Symmetric matrices: real eigenvalues

The General Case

$$A\mathbf{u} = \lambda\mathbf{u} \quad (= \mathbf{u} \cdot \lambda)$$

$$AU = U\Lambda \quad \Rightarrow \quad A = U\Lambda U^{\mathrm{T}}$$

$U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$: orthogonal (orthonormal)
$\mathbf{u}_1, \ldots, \mathbf{u}_n$: basis for $\mathbb{R}^n$

The Power Method Demonstrated

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$\mathbf{v}^{(1)} = [1, 2]^{\mathrm{T}}$, $\mathbf{v}^{(2)} = A\mathbf{v}^{(1)}$, etc.: $\mathbf{v}^{(i)} = A^i\mathbf{v}$:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 13 \\ 14 \end{bmatrix}, \ldots, \begin{bmatrix} 29524 \\ 29525 \end{bmatrix}, \begin{bmatrix} 88573 \\ 88574 \end{bmatrix}, \ldots$$

For large i : $\mathbf{v}^{(i)} = A\mathbf{v}^{(i-1)} = 3\mathbf{v}^{(i-1)}$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \lambda_1 = 3$$

Power Method Explained

A has eigenvectors $[\mathbf{u}_1, \mathbf{u}_2]$, eigenvalues $\lambda_1 > \lambda_2$

$$\mathbf{v}^{(1)} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$$

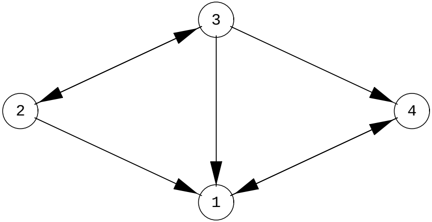
$$\mathbf{v}^{(i)} = A\mathbf{v}^{(i-1)} = c_1\lambda_1^{i-1}\mathbf{u}_1 + c_2\lambda_2^{i-1}\mathbf{u}_2$$

λ_1 dominates.

Case Study: Google

web connectivity matrix

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$



Page Ranking I

l_j : number of links that page j links to: $[1, 2, 3, 1]$.
Normalize:

$$D = \begin{bmatrix} 0 & 1/2 & 1/3 & 1 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1/3 & 0 \end{bmatrix}.$$

r_j: rank of page j depends on no. of (weighted) links *to it*.

$$r_1 = \frac{1}{2}r_2 + \frac{1}{3}r_3 + r_4, \quad r_2 = \frac{1}{3}r_3, \quad r_3 = \frac{1}{2}r_2, \quad r_4 = r_1 + \frac{1}{3}r_3$$

Page Ranking II

$$\mathbf{r} = D\mathbf{r}$$

$\mathbf{r}$ = eigenvector to D's (dominant) eigenvalue 1.

$$\mathbf{r} = \begin{bmatrix} 0.71 \\ 0 \\ 0 \\ 0.71 \end{bmatrix}$$

Jacobi iteration

A : 2×2 symmetric

$$\Lambda = U^{\mathrm{T}} A U.$$

U : rotation matrix, U^{T} : inverse rotation.

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

$$\tan 2\alpha = \frac{2a_{1,2}}{a_{1,1} - a_{2,2}}.$$

Jacobi Example

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$\tan 2\alpha = 4/0 = \infty \Rightarrow \alpha = 45^\circ$.

Jacobi cont'd

$$U = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}.$$

$$U^{\mathrm{T}} A U = \begin{bmatrix} 2.999 & 0 \\ 0 & 0.999 \end{bmatrix},$$

$\lambda_{1,2} = 2.999, 0.999.$

Jacobi General

$$U = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\alpha & \cdots & -\sin\alpha & & \\ & & \vdots & \ddots & \vdots & & \\ & & \sin\alpha & \cdots & \cos\alpha & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

$a'_{i,j} = a_{j,i} = 0.$

Eigenvalues via Determinants

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$[A - \lambda I]\mathbf{v} = \mathbf{0}$$

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \text{zero of a polynomial}$$

Example

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 4 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 4-\lambda & 2 \\ 1 & 2 & -3-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + 17\lambda - 21.$$

$$\lambda_{1,2,3} = 4.835, -3.755, 2.919$$

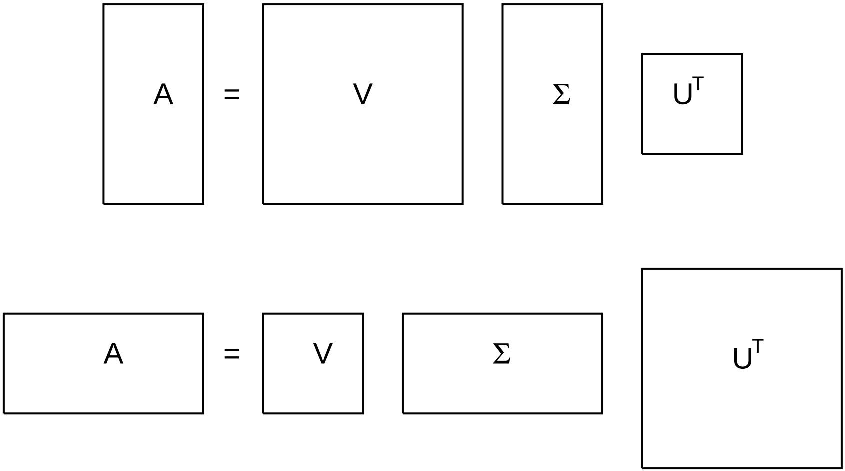
SVD

$$AU = V\Sigma$$

U, V : orthogonal

$$A = V\Sigma U^{-1} = V\Sigma U^{\mathrm{T}}$$

SVD



Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \qquad A^{\mathrm{T}}A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Finding the SVD

$$AU = V\Sigma$$
$$(AU)^{\mathrm{T}}AU = (V\Sigma)^{\mathrm{T}}V\Sigma$$
$$U^{\mathrm{T}}A^{\mathrm{T}}AU = \Sigma^{\mathrm{T}}V^{\mathrm{T}}V\Sigma$$
$$A^{\mathrm{T}}A = U\Lambda U^{\mathrm{T}}$$

U, Λ : Eigenvectors/values of $A^{\mathrm{T}}A$.

Condition Number

$$\sigma_1/\sigma_n$$

$$A = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.7 & 0.24 & 0.06 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$\sigma_{1,2,3} = 1.3, 0.26, 0.006$$

Diagonal Pseudoinverse

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \qquad D^+ = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}, \qquad DD^+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Pseudoinverse

$$A^{+} = V\Sigma^{+}U^{\mathrm{T}}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^{+} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \\ 0 & -0.25 \end{bmatrix}.$$

Linear Systems

$$A\mathbf{u} = \mathbf{b}$$

$$\mathbf{u} = A^{+}\mathbf{b}$$

PCA

$$A = V\Sigma U^{\mathrm{T}} = \sigma_1 \mathbf{v}_1 \mathbf{u}_1^{\mathrm{T}} + \sigma_2 \mathbf{v}_2 \mathbf{u}_2^{\mathrm{T}} + \ldots$$

Truncate for small σ_i: approximate A.

Case Study: Eigenfaces

Face image: 100×100 pixels $\Rightarrow$ 10,000-dim. vector $\mathbf{v}$
Gallery of 1,000 images $\mathbf{v}_i$
New face $\mathbf{v}$ in Gallery?

$$\|\mathbf{v}_i - \mathbf{v}\| < \epsilon; \qquad i = 1, \ldots, 1,000$$

Expensive!

Eigenfaces

Approximate each $\mathbf{v}_i$ by sum of 100 vectors $\mathbf{e}_i$:

$$\mathbf{v}_i \approx \sum_{j=1}^{100} s_{i,j}\mathbf{e}_j.$$

Eigenfaces



Eigenfaces

For new $\mathbf{v}$, find s_j:

$$\mathbf{v} \approx \sum_{j=1}^{100} s_j \mathbf{e}_j,$$

Check

$$\|\mathbf{s} - \mathbf{s}_i\| < \epsilon; \qquad i = 1, \ldots, 1000$$

Cheap!

Determinants

$$|A| = \sigma_1 \cdot \ldots \cdot \sigma_n$$