Data Fitting

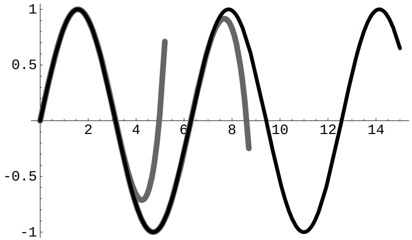
Gerald Farin, Dianne Hansford
Arizona State University

August 30, 2008

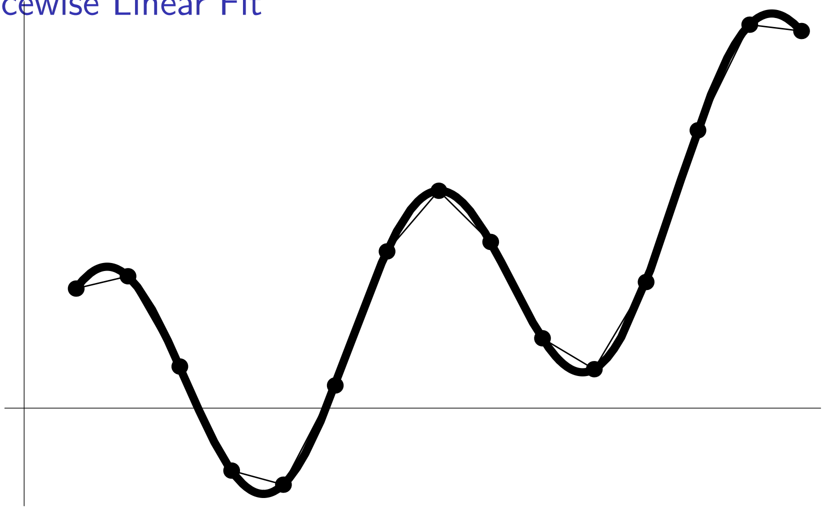
Taylor Expansion

$$p(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \ldots$$

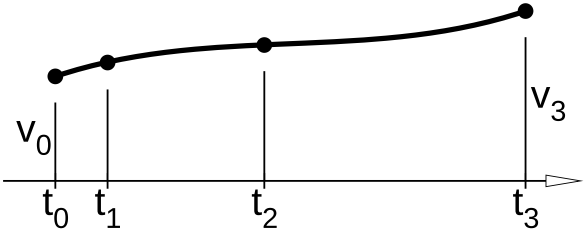
$$\ldots + \frac{1}{n!} f^{(n)}(0) x^n$$



Piecewise Linear Fit



Cubic Fit



Cubic Fit

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Data: $(t_0, v_0), \ldots, (t_3, v_3)$

Conditions:

$$\begin{aligned} p(t_0) &= v_0 \\ p(t_1) &= v_1 \\ p(t_2) &= v_2 \\ p(t_3) &= v_3 \end{aligned}$$

Cubic Fit

$$
\begin{aligned}
v_0 &= a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 \\
v_1 &= a_0 + a_1 t_1 + a_2 t_1^2 + a_3 t_1^3 \\
v_2 &= a_0 + a_1 t_2 + a_2 t_2^2 + a_3 t_2^3 \\
v_3 &= a_0 + a_1 t_3 + a_2 t_3^2 + a_3 t_3^3.
\end{aligned}
$$

Matrix Form:

$$
\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.
$$

Abbreviated:

$$
\mathbf{v} = T\mathbf{a}.
$$

Least Squares, Polynomial

$$p(t) = a_0 + a_1 t + \ldots + a_n t^n$$

$$\begin{aligned} p(t_0) &= v_0 \\ &\vdots \\ p(t_L) &= v_L \end{aligned}$$

$$\begin{aligned} p(t_0) &= a_0 + a_1 t_0 + \ldots + a_n t_0^n \\ &\vdots \\ p(t_L) &= a_0 + a_1 tL + \ldots + a_n t_L^n. \end{aligned}$$

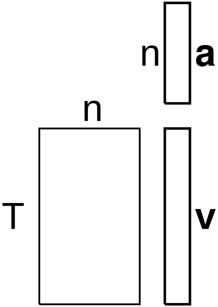
Matrix form:

$$\mathbf{v} = T\mathbf{a}$$

Least Squares

Solution:

$$T^{\mathrm{T}}\mathbf{v} = T^{\mathrm{T}}T\mathbf{a}$$



Weights and Noise

Replace

$$p(t_i) = a_0 + a_1 t_i + \ldots + a_n t_i^n$$

by

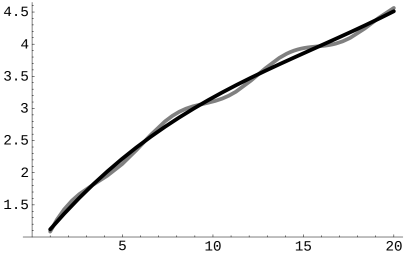
$$w_i p(t_i) = w_i a_0 + w_i a_1 t_i + \ldots + w_i a_n t_i^n$$

w_i high for high-confidence data

Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

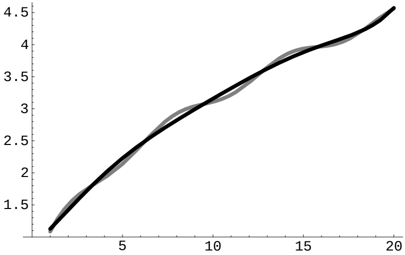
$$n = 3$$



Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

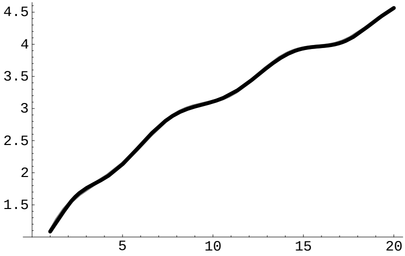
$$n = 6$$



Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

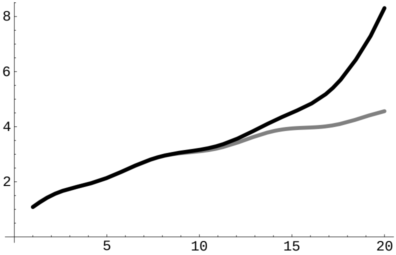
$$n = 10$$



Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

$$n = 11$$

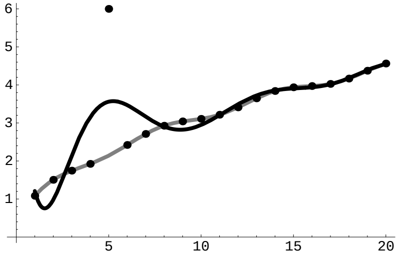


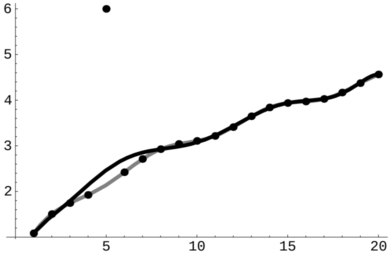
Outliers

1. compute standard fit w/o weights.
2. Then set

$$w_i = \frac{1}{|p(t_i) - v_i| + 0.1}$$

Outliers, Example





General Case

$$f(t) = c_0 B_0(t) + \ldots + c_n B_n(t)$$

$B_0, \ldots, B_n$: arbitrary basis functions

Analagous normal equations

A Stable Basis

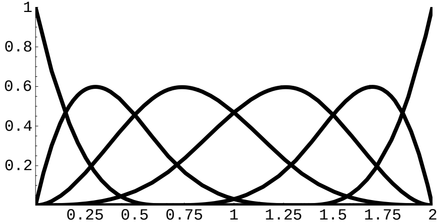
$$B_j^n(t) = \binom{n}{j} \frac{(t_L - t)^{n-j}(t - t_0)^j}{(t_L - t_0)^n}; \quad i = 0, \ldots, n$$

No problem with $n = 11$ case above

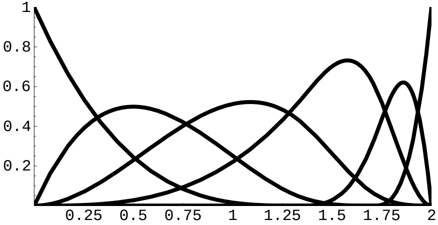
B-splines N_i^3

- piecewise cubic
- pieces over knot sequence $u_0, \ldots, u_L$
- C^2
- local support

Examples



1
0.8
0.6
0.4
0.2
0.25 0.5 0.75 1 1.25 1.5 1.75 2



1
0.8
0.6
0.4
0.2
0.25 0.5 0.75 1 1.25 1.5 1.75 2

Interpolation

spline function:

$$s(t) = \sum_{i=0}^{K+2} N_i^3(t) d_i$$

Given $(v_0, y_0), \ldots, (v_{K+2}, y_{K+2})$
Find $d_0, \ldots, d_{K+2}$ from

$$\begin{bmatrix} N_0^3(v_0) & \cdots & N_{K+2}^3(v_0) \\ \vdots & & \vdots \\ N_0^3(v_{K+2}) & \cdots & N_{K+2}^3(v_{K+2}) \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_{K+2} \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_{K+2} \end{bmatrix}$$

Matrix: mostly 0 entries

Choice of v_i

$u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5$

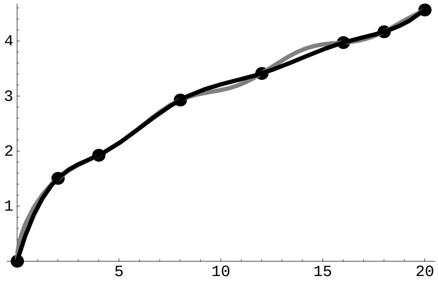
$v_0 \; v_1 \; v_2 \quad v_3 \quad v_4 \quad v_5 \; v_6 \; v_7$

Example

knots:

$$0, 2, 4, 8, 12, 16, 18, 20$$

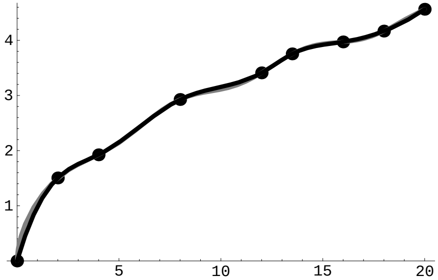
$$v_1 = (u_0 + u_1)/2, \quad v_6 = (u_4 + u_5)/2$$



Example

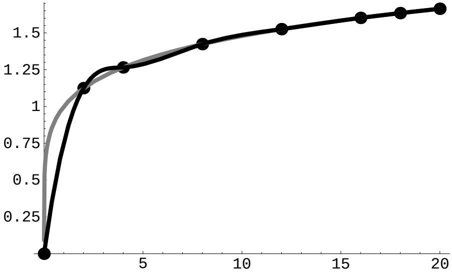
knots:

$$0, 2, 4, 8, 12, \mathbf{13.5}, 16, 18, 20$$



Oscillations

convex data
nonconvex interpolant



Least Squares

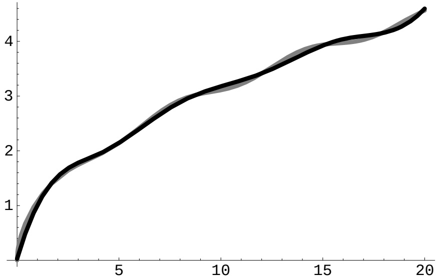
Knots: $u_0, \ldots, u_K$
data at: $(t_i, v_i); j = 0, \ldots, L$ (with $L > K$)
linear system:

$$\begin{bmatrix} N_0^3(t_0) & \cdots & N_{K+2}^3(t_0) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ N_0^3(t_L) & \cdots & N_{K+2}^3(t_L) \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_{K+2} \end{bmatrix} = \begin{bmatrix} v_0 \\ \vdots \\ \vdots \\ v_L \end{bmatrix}$$

Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

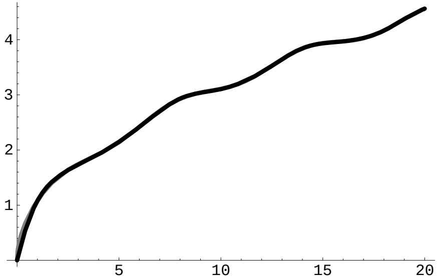
data at: $t_1 = 1, \ldots, t_{20} = 20$
knot sequence:
$\{0, 4, 8, 12, 16, 20\}$



Example

$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

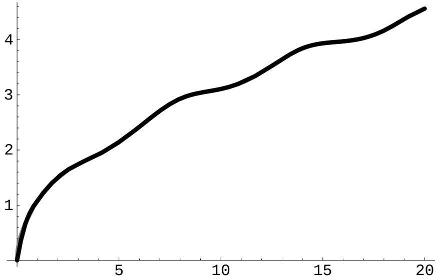
data at: $t_1 = 1, \ldots, t_{20} = 20$
knot sequence:
$\{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$



Example

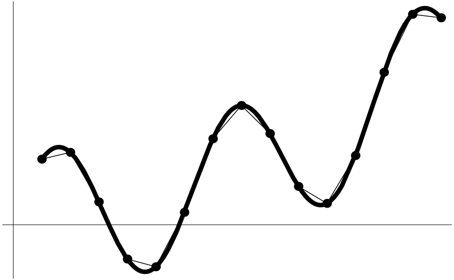
$$f(t) = \sqrt{t} + 0.1 \sin t, \quad t \in [1, 20]$$

data at: $t_1 = 1, \ldots, t_{20} = 20$
knot sequence:
$\{0, \mathbf{1}, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$



Integrals

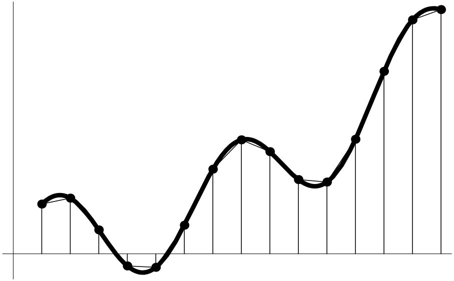
pw linear data fit:
use to find areas



Trapezoidal rule

area:

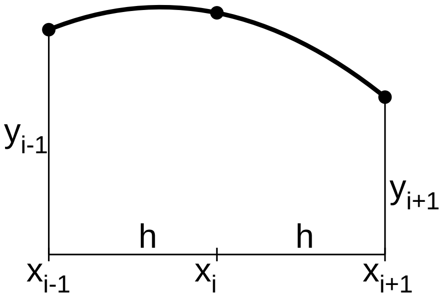
$$A_i = (x_{i+1} - x_i)\frac{y_i + y_{i+1}}{2}.$$



Simpson's rule

area:

$$S_i = \frac{h}{3}[y_{i-1} + 4y_i + y_{i+1}]$$



Derivatives

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$