Linear Systems

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Mixing Chemicals

	Container 1	Container 2	Container 3
A	40 g	40 g	70 g
B	30 g	80 g	60 g
C	60 g	10 g	30 g

new mixture: 50 grams each of A, B, and C. Fractions f_1, f_2, f_3 of A, B, C:

$$\begin{aligned} 40f_1 + 40f_2 + 70f_3 &= 50, \\ 30f_1 + 80f_2 + 60f_3 &= 50, \\ 60f_1 + 10f_2 + 30f_3 &= 50. \end{aligned}$$

$$\begin{bmatrix} 40 & 40 & 70 \\ 30 & 80 & 60 \\ 60 & 10 & 30 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.22 \\ 0.19 \end{bmatrix},$$

Thus 70% of the contents of container 1, 22% of container 2, and 19% of container 3.

Linear Systems

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$A\mathbf{u} = \mathbf{b}.$$

$$\mathbf{u} = A^{-1}\mathbf{b},$$

Do not use inverse!

Gauss Elimination I

$$\begin{aligned} 2u_1 - 2u_2 &= \quad 10, \\ 6u_1 + 4u_2 &= \quad 10, \end{aligned}$$

-3 * first eq. added to second eq.:

$$\begin{aligned} 2u_1 - 2u_2 &= \quad 10 \\ 10u_2 &= \ -20. \end{aligned}$$

Thus $u_2 = -2$.

Gauss Elimination II

Transform

$$\begin{bmatrix} 2 & -2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

to

$$\begin{bmatrix} 2 & -2 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -20 \end{bmatrix}$$

Gauss Elimination III

$$\begin{bmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix} \Rightarrow \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix} \Rightarrow \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{bmatrix} \Rightarrow \begin{bmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{bmatrix}$$

diagonal $\star$s nonzero!

Gauss Elimination IV

$$\begin{bmatrix} 2 & -2 & 0 \\ 6 & 4 & -1 \\ 8 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ -10 \end{bmatrix}$$

transformed to

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 10 & -1 \\ 0 & 10 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -20 \\ -50 \end{bmatrix}.$$

No solution!

Stability

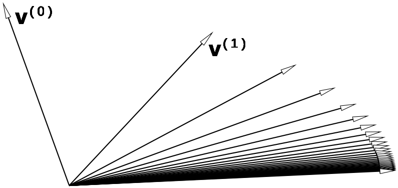
$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.7 & 0.24 & 0.06 \\ 0.6 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 9.49 \\ 8.856 \\ 8.16 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} 10.2 \\ 5.1 \\ 8.2 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.7 & 0.24 & 0.06 \\ 0.6 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 9.69 \\ 8.856 \\ 8.16 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} 12.6 \\ 1.5 \\ -5.4 \end{bmatrix}$$

System is **ill-conditioned**

Vector Sequences

$$\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots$$



Vector Norms

Euclidean Norm

$$\|\mathbf{w}\| = \sqrt{\mathbf{w} \cdot \mathbf{w}}.$$

Norms

- $\|\mathbf{w}\| > 0$ when $\mathbf{w} \neq \mathbf{0}$,
- $\|\mathbf{w}\| = 0$ when $\mathbf{w} = \mathbf{0}$,
- $\|c\mathbf{w}\| = |c|\|\mathbf{w}\|$ for any scalar c,
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ (triangle inequality)

Iterative Solving I

$$\begin{bmatrix} 4 & 1 & 0 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \text{guess}: \quad \mathbf{u}^{(1)} = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A\mathbf{u}^{(1)} \neq \mathbf{b}$$

Iterative Solving II

$$\begin{aligned} 4u_1^{(2)} + 1 &= 1, \\ 2 + 5u_2^{(2)} + 1 &= 0, \\ -1 + 2 + 4u_3^{(2)} &= 3, \end{aligned}$$

$$\mathbf{u}^{(2)} = \begin{bmatrix} 0 \\ -0.6 \\ 0.5 \end{bmatrix}.$$

$$
\begin{aligned}
4u_1^{(3)} - 0.6 &= 1, \\
5u_2^{(3)} + 0.5 &= 0, \\
-1.2 + 4u_3^{(3)} &= 3,
\end{aligned}
$$

$$
\mathbf{u}^{(3)} = \begin{bmatrix} 0.4 \\ -0.1 \\ 1.05 \end{bmatrix} \qquad \mathbf{u}^{(\infty)} = \begin{bmatrix} 0.3333 \\ -0.3333 \\ 1.0 \end{bmatrix}
$$

Gauss-Jacobi I

$$A\mathbf{u} = \mathbf{b} \quad \text{guess :} \quad \mathbf{u}^{(1)}$$

$$A = D + R; \qquad d_{i,i} = a_{i,i}, \quad \text{other} \quad d_{i,j} = 0.$$

$$D\mathbf{u} + R\mathbf{u} = \mathbf{b} \quad \text{or} \quad \mathbf{u} = D^{-1}[\mathbf{b} - R\mathbf{u}].$$

Iteration:

$$\mathbf{u}^{(k+1)} = D^{-1}[\mathbf{b} - R\mathbf{u}^{(k)}],$$

Gauss-Jacobi II

Convergence:

$$\|A\mathbf{u}^{(k)} - \mathbf{b}\| \quad \rightarrow 0 \ \text{ for } \ k \rightarrow \infty \ \text{ ?}$$

Yes if diagonal elements of A are large:

$$|a_{i,i}| > |a_{i,1}| + \ldots + |a_{i,n}| \ \text{ w/o } \ |a_{i,i}|$$

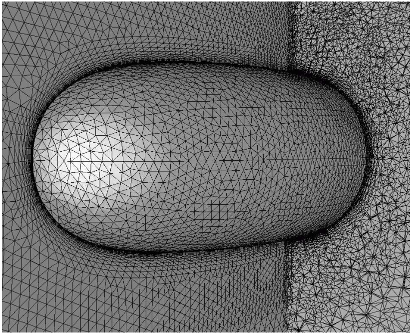
Then: A is **diagonally dominant**

Case Study: Fluid Flow

$\mathbf{x}_i$: mesh vertex, $\mathbf{y}_j$: its neighbors
FEM equations:

$$\mathbf{x}_i = \alpha_1 \mathbf{y}_1 + \ldots + \alpha_N \mathbf{y}_N$$

sparse matrix, solve w/
Gauss-Jacobi



Overdetermined Systems

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

$$A\mathbf{u} = \mathbf{b}$$

A is not square!
Solution (normal equations):

$$A^{\mathrm{T}}A\mathbf{u} = A^{\mathrm{T}}\mathbf{b}.$$

Example: Linear Fit

Given: $(1, 1), (2, 0), (3, 4), (4, 4), (5, 6)$
Wanted: $y = ax + b$ linear fit.

$$\begin{bmatrix} 1 \\ 0 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Normal equations:

$$\begin{bmatrix} 59 \\ 15 \end{bmatrix} = \begin{bmatrix} 55 & 15 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

solution:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.4 \\ -1.2 \end{bmatrix}$$