Dynamical Processes

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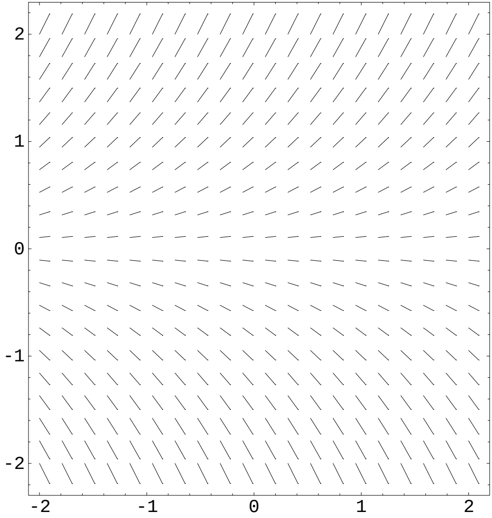
August 24, 2008

Population Growth

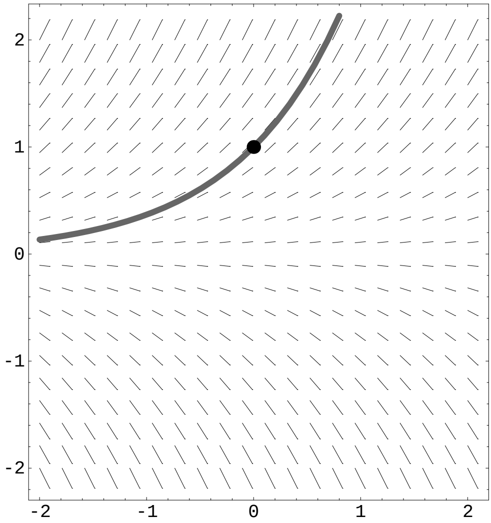
Malthus:

$$p'(t) = c \cdot p(t)$$

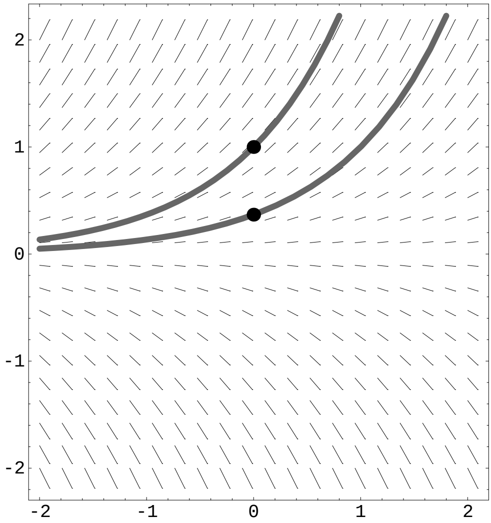
Slope Fields



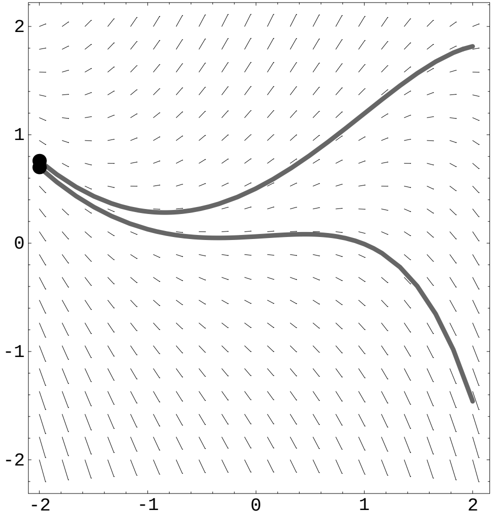
Slope Fields



Slope Fields



A Different Slope Field



Recap

ODE:

$$y'(x) = f(x, y)$$

f: slope field, assign a slope y' to each x, y

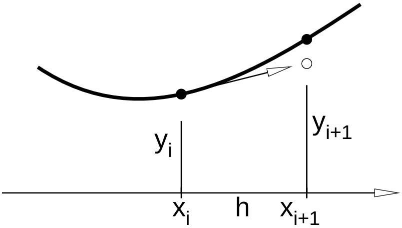
Euler's Method

Tangent:

$$t(x) = y(x_0) + f(x_0, y_0) \cdot (x - x_0).$$

Step length h: sequence $x_0, x_1, x_2, \ldots$, with

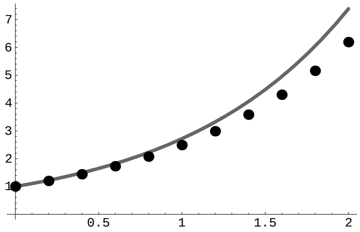
$$x_i = x_0 + ih, \quad \text{find} \quad y_i = y(x_i)$$



Example: 10 points

$$y' = y$$

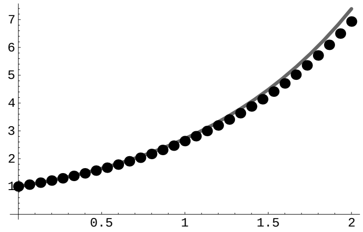
$$y(0) = 1$$



Example: 30 points

$$y' = y$$

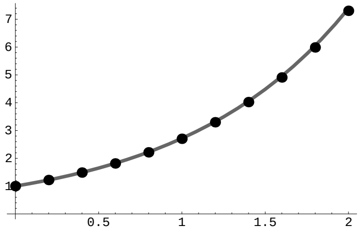
$$y(0) = 1$$



Heun's Method

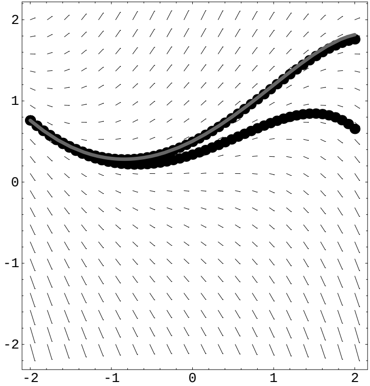
$$y_{i+1} = y_i + \frac{1}{2}h(f_i + f_{i+1}).$$

$$y' = y, \quad y(0) = 1$$



Heun *vs* Euler

$$f(x, y) = y - 0.4x^2$$

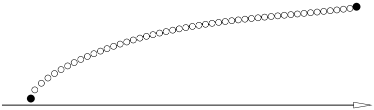


Finite Differences

Boundary Value Problem:

$$y''(x) = p(x)y'(x)+q(x)y(x)+r(x)$$

$y(a) = y_0$ and $y(b) = y_n$

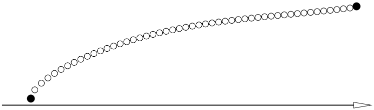


Finite Differences

Discretize:

$$y'(x_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

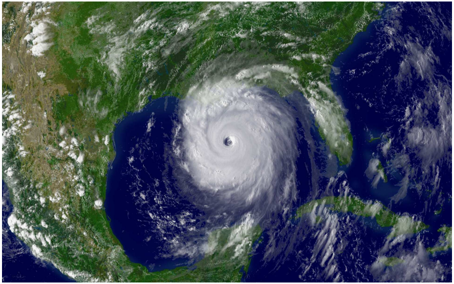
$$y''(x_i) \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

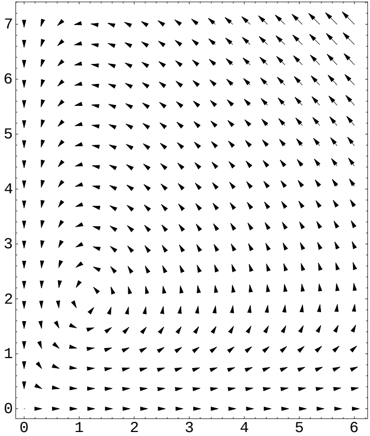


Finite Difference System

$$
\begin{bmatrix}
b_1 & c_1 & & & & \\
a_2 & b_2 & c_2 & & & \\
 & a_3 & b_3 & c_3 & & \\
 & & & \ddots & & \\
 & & & a_{n-2} & b_{n-2} & c_{n-2} \\
 & & & & a_{n-1} & b_{n-1}
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-2} \\ y_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
h^2 r_1 - a_1 y_0 \\ h^2 r_2 \\ h^2 r_3 \\ \vdots \\ h^2 r_{n-2} \\ h^2 r_{n-1} - c_{n-1} y_n
\end{bmatrix}
$$

Vector Fields





Dynamical Systems

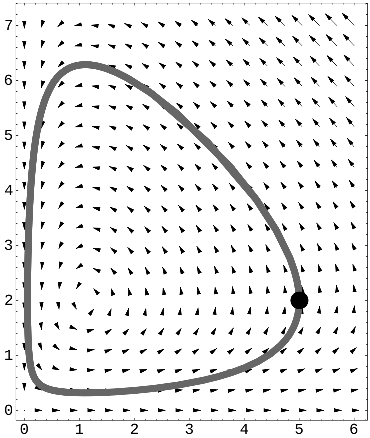
$f(t)$: Foxes at time t
$r(t)$: Rabbits at time t

$$\begin{aligned} r'(t) &= \quad r(t) - f(t)r(t) \\ f'(t) &= -f(t) + f(t)r(t) \end{aligned}$$

Foxes and Rabbits

Refined equations:

$$\begin{aligned} r'(t) &= \quad 2r(t) - 0.5f(t)r(t) \\ f'(t) &= \quad -f(t) + 0.9f(t)r(t) \end{aligned}$$



General Systems

$$\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1(t), \ldots, x_n(t)) \\ \vdots \\ f_n(t, x_1(t), \ldots, x_n(t)) \end{bmatrix}$$

short:

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t))$$

Case Study: The Lorenz Attractor

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$[x(t), y(t), z(t)]$: path of a particle

$$\begin{aligned} x'(t) &= a(y(t) - x(t)), \\ y'(t) &= x(t)(b - z(t)) - y(t), \\ z'(t) &= x(t)y(t) - cz(t) \end{aligned}$$

Lorenz