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ScienceDirect
Computer Aided Geometric Design 25 (2008) 80–95
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

On the approximation order of tangent estimators

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Received 2 February 2007; received in revised form 23 May 2007; accepted 24 May 2007

Available online 2 June 2007

Abstract

A classic problem in geometric modelling is curve interpolation to data points. Some of the existing interpolation schemes only require point data, whereas others, require higher order information, such as tangents or curvature values, in the data points. Since measured data usually lack this information, estimation of these quantities becomes necessary. Several tangent estimation methods for planar data points exist, usually yielding different results for the same given point data. The present paper thoroughly analyses some of these methods with respect to their approximation order. Among the considered methods are the classical schemes FMILL, Bessel, and Akima as well as a recently presented conic precision tangent estimator. The approximation order for each of the methods is theoretically derived by distinguishing purely convex point configurations and configurations with inflections. The approximation orders vary between one and four for the different methods. Numerical examples illustrate the theoretical results.

1. Introduction

A classic problem in geometric modelling is curve interpolation to data points. Different interpolation schemes exist, see, e.g., Farin (2001). Some of them only require the point data, whereas others, especially the piecewise interpolation schemes, require higher order information, such as tangents or curvature values, in the data points. Since measured data usually lack this information, estimation of these quantities becomes necessary.

This paper deals with the problem of estimating tangents in given planar data points. There exist different methods for this purpose. We compare five schemes, that are popular in the CAGD area, first with respect to their practical performance, and then we thoroughly investigate their respective approximation orders. The considered tangent estimators are FMILL's method, Bessel's and Akima's method, see Farin (2001), as well as a circle based and a general conic based method. Conic based tangent estimators are, e.g., used in Schaback (1993), Liming (1944), Pavlidis (1983).

FMILL's, Bessel's and the circle based method take three consecutive points as input data, and the tangent is estimated in the middle point. FMILL simply produces a line parallel to the one joining the first and the third point,

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0167-8396/$ – see front matter
doi:10.1016/j.cagd.2007.05.005

Bessel takes the tangent to a parametric parabola interpolating the three points, and the circle based method uses the tangent to the circle interpolating the three data points. Akima's method and the conic based method take five consecutive points estimating the tangent in the third point. As Bessel's method also Akima's method needs parameter values in the data points to be estimated, and obtains the tangent vector by a certain weighted combination of the five data points. The more recent conic based method (Albrecht et al., 2005) produces the tangent to the interpolating conic of the given five points. It uses a simple algorithm that is based on a theorem from classical projective geometry, so-called Pascal's theorem, and thus necessitates very few computations to achieve this tangent with conic precision, in contrast to computing the implicit conic.

Regarding the comparison of these five methods, experiments clearly illustrate a better performance of the conic based method with respect to the other schemes showing a high suitability for applications in the area of convexity preserving interpolation. We determine the approximation orders of all five considered tangent estimation methods, and thus prove that for convex configurations the conic based method is of approximation order four, whereas the circle based method has approximation order two, and FMILL's, Bessel's and Akima's in general only have approximation order one. In the case of inflection points we obtain slightly different results.

The paper is organized as follows. In Section 2 we first recall the different tangent estimation schemes (Sections 2.1 and 2.2), and then apply them to a set of sample curves comparing their performance (Section 2.3). Section 3 is dedicated to a theoretical determination of the approximation orders of all involved methods (Section 3.1) followed by numerical experiments illustrating the theoretical results (Section 3.2). Section 4 contains some concluding remarks.

2. Current tangent estimators

2.1. Classical methods: FMILL, Bessel, Akima, Circle

The most popular schemes for tangent estimation are FMILL, Bessel's, and Akima's method, see, e.g., Farin (2001), as well as the tangent to the circle through three consecutive points.

FMILL's method (also known as a Catmull–Rom spline (Catmull and Rom, 1974)) calculates a tangent vector $\mathbf{v} \in \mathbb{R}^2$ in $\mathbf{p}_3$ as

$$\mathbf{v} = \mathbf{p}_4 - \mathbf{p}_2. \tag{1}$$

Bessel's method calculates a tangent vector $\mathbf{v} \in \mathbb{R}^2$ in $\mathbf{p}_3$ to the interpolating parabola through the points $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ with corresponding parameter values τ_2, τ_3, τ_4. This yields

$$\mathbf{v} = \frac{1-\alpha}{\Delta_2}(\mathbf{p}_3 - \mathbf{p}_2) + \frac{\alpha}{\Delta_3}(\mathbf{p}_4 - \mathbf{p}_3), \tag{2}$$

where $\Delta_i = \tau_{i+1} - \tau_i$, $\alpha = \frac{\Delta_2}{\Delta_2 + \Delta_3}$.

Akima's method obtains a tangent vector $\mathbf{v} \in \mathbb{R}^2$ in $\mathbf{p}_3$ as follows by using the five points $\mathbf{p}_1, \ldots, \mathbf{p}_5$ with corresponding parameter values $\tau_1, \ldots, \tau_5$.

$$\mathbf{v} = (1-\alpha)\mathbf{a}_2 + \alpha \mathbf{a}_3, \tag{3}$$

where $\mathbf{a}_i = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{\tau_{i+1} - \tau_i}$, $\alpha = \frac{\|\Delta \mathbf{a}_1\|}{\|\Delta \mathbf{a}_1\| + \|\Delta \mathbf{a}_3\|}$, $\Delta \mathbf{a}_i = \mathbf{a}_{i+1} - \mathbf{a}_i$. Chord length parameterization of the data points was used for the implementation of Bessel and Akima tangents. Further details on these schemes may be found in Farin (2001).

The tangent at $\mathbf{p}_3$ to the circle through the points $\mathbf{p}_2(x_2, y_2)$, $\mathbf{p}_3(0, 0)$, $\mathbf{p}(x_4, y_4)$ has the normal

$$\mathbf{n} = \begin{pmatrix} x_4^2 y_2 - x_2^2 y_4 + y_2 y_4^2 - y_2^2 y_4 \\ x_2^2 x_4 - x_4^2 x_2 + x_4 y_2^2 - x_2 y_4^2 \end{pmatrix}. \tag{4}$$

2.2. Conic method

2.2.1. Geometric background

First, we give our nomenclature for geometric entities and operations on them. A point $\bar{\mathbf{p}}$ in projective space $\mathbb{P}^2$ is represented with lowercase letters and its corresponding affine point $\mathbf{p}$ in $\mathbb{E}^2$ is given, respectively as

$$\bar{\mathbf{p}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} p_1/p_0 \\ p_2/p_0 \end{bmatrix}.$$

Points with $p_0 = 0$ are called *points at infinity*. A line in projective space $\mathbb{P}^2$ is represented with capital letters as $\bar{\mathbf{L}} = [L_0, L_1, L_2]$, where the coordinates are the coefficients for the implicit equation of the line $L_0 + L_1 x + L_2 y = 0$. Furthermore, we will use the notation that the line $\bar{\mathbf{L}}_{ij}$ is the *join* between the points $\bar{\mathbf{p}}_i$ and $\bar{\mathbf{p}}_j$.

The *principle of duality* in projective geometry allows for the joining of points and the intersection of lines to be achieved with a cross product. Thus the join between two points, $\bar{\mathbf{L}} = \bar{\mathbf{p}} \wedge \bar{\mathbf{q}}$, results in a line; Likewise, the intersection of two lines, $\bar{\mathbf{p}} = \bar{\mathbf{L}} \wedge \bar{\mathbf{M}}$, results in a point. More information on projective geometry as it relates to Computer Aided Geometric Design (CAGD) may be found in Farin (1999).

Theorem 1 *(Pascal's). Let $\bar{\mathbf{p}}_1, \ldots, \bar{\mathbf{p}}_6$ be six distinct points on a non-degenerate conic section. Divide the points into two ordered sets of three points, for example $(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_5, \bar{\mathbf{p}}_3)$ and $(\bar{\mathbf{p}}_4, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_6)$. Then, the intersection points*

$$\bar{\mathbf{a}} = \bar{\mathbf{L}}_{12} \wedge \bar{\mathbf{L}}_{54}, \qquad \bar{\mathbf{b}} = \bar{\mathbf{L}}_{56} \wedge \bar{\mathbf{L}}_{32}, \qquad \bar{\mathbf{c}} = \bar{\mathbf{L}}_{16} \wedge \bar{\mathbf{L}}_{34},$$

lie on a line called the Pascal line.

For an illustration see Fig. 1. Any partitioning of the points will result in a Pascal line, thus there are many such lines.

For our tangent estimation scheme, we want to have $\bar{\mathbf{p}}_3$ and $\bar{\mathbf{p}}_4$ from above coalesce to one point. Now, re-label $\bar{\mathbf{p}}_5$ as $\bar{\mathbf{p}}_4$ and $\bar{\mathbf{p}}_6$ as $\bar{\mathbf{p}}_5$. Thus, according to Pascal's theorem, the points

$$\bar{\mathbf{a}} = \bar{\mathbf{L}}_{12} \wedge \bar{\mathbf{L}}_{34}, \qquad \bar{\mathbf{b}} = \bar{\mathbf{L}}_{54} \wedge \bar{\mathbf{L}}_{32}, \qquad \bar{\mathbf{c}} = \bar{\mathbf{L}}_{15} \wedge \bar{\mathbf{L}}_{33}, \tag{5}$$

are collinear, where $\bar{\mathbf{L}}_{33}$ is the tangent at $\bar{\mathbf{p}}_3$ (see, e.g., Pascal (1910, p. 236) or Chasles (1865, p .43). This is illustrated in Fig. 2.

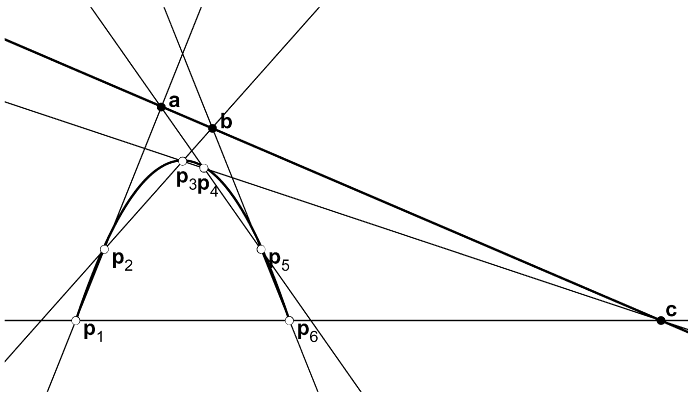


Fig. 1. Pascal's theorem: points $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ are collinear.

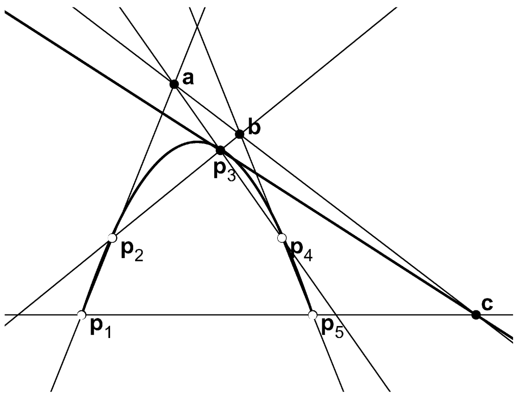


Fig. 2. Tangent construction: tangent $\bar{\mathbf{L}}_{33}$ at $\bar{\mathbf{p}}_3$ is formed using a special application of Pascal's theorem where two points have coalesced.

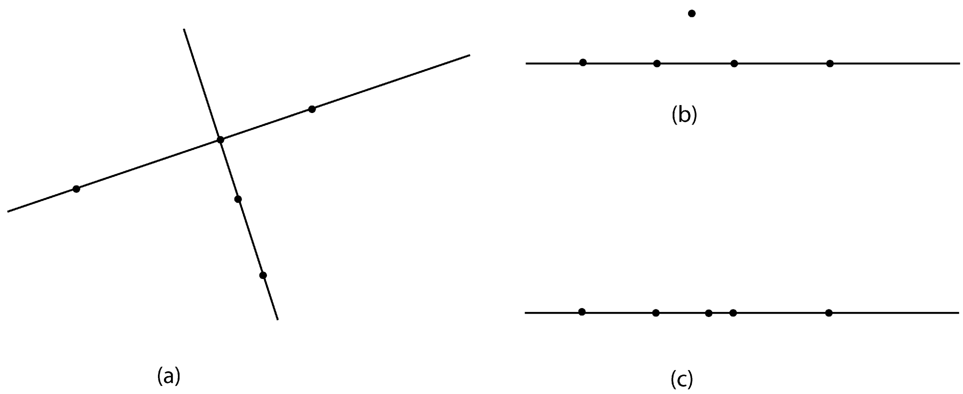


Fig. 3. Degenerate configurations, where the algorithm (6) produces an invalid result: (a) the conic degenerates into a pair of intersecting lines, where each of the lines contains 3 data points, the intersection point of the lines being a data point, (b) the conic degenerates into a pair of intersecting lines, where one of the lines contains 4 data points, (c) the conic degenerates into a double line containing all five data points.

2.2.2. *Algorithm*

Based on the lines and points in (5), the tangent $\bar{\mathbf{L}}_{33}$ at $\bar{\mathbf{p}}_3$ is constructed as

$$\bar{\mathbf{L}}_{33} = \bar{\mathbf{p}}_3 \wedge \big(\bar{\mathbf{L}}_{15} \wedge (\bar{\mathbf{a}} \wedge \bar{\mathbf{b}})\big). \tag{6}$$

By appropriate relabeling, this algorithm may be adjusted to find the tangent at each of the five points.[1]

An alternative approach using cross ratios of lines is discussed in Albrecht et al. (2005).

If the underlying conic degenerates into a pair of intersecting lines or a double line, the algorithm fails to produce a result in the configurations illustrated in Fig. 3, where $\bar{\mathbf{p}}_3$ can be any of the five marked points. If the conic degenerates into a pair of lines, where the intersection point of the lines is *not* a data point the algorithm correctly produces the line containing the point $\bar{\mathbf{p}}_3$ as estimated tangent.

2.3. *Examples*

We tested the above tangent estimation methods by measuring the angle ε in radians between the exact tangent and the estimates on a set of sample curves.

A representative set of the curves tested, along with the points selected from them, are given in Table 1 and illustrated in Fig. 4. Results from the tests, obtained with the computer algebra system MAPLE, are given in Table 2.

These examples clearly illustrate that for planar, convex data, the method with conic precision performs better than the standard tangent estimation methods FMILL, Bessel's, Akima's and the circle method. For non-convex data all of these methods may perform very badly as is illustrated in Fig. 5; here, the points are taken from a degree 4 polynomial parametric curve and the interpolating hyperbola of the conic method, the interpolating parabola of Bessel's method,

Table 1
Curves tested: the curve type, equation, and parameter values extracted are listed

Curve	$M(x(t), y(t))$	$(t_1, t_2, t_3, t_4, t_5)$
1: Polynomial	$\big(t, \frac{1}{5}(1-(1-t)^5)\big)$	$(0.1, 0.2, 0.3, 0.5, 0.8)$
2: Witch of Agnesi	$\big(t, \frac{1}{1+t^2}\big)$	$(-2, -1.9, -1.7, -1.5, -1)$
3: Folium of Descartes	$\big(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\big)$	$(0.1, 0.25, 0.5, 0.85, 0.9)$
4: Bicorn	$\big(\sin t, \frac{\cos^2 t}{2-\cos t}\big)$	$\big(\frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{5}, 0.9\big)$
5: Tear Drop curve	$\big(\cos t, \sin t . \sin^2\big(\frac{t}{2}\big)\big)$	$(1.8, 1.85, 1.9, 1.95, 2.2)$
6: Exponential	$(t, \exp t)$	$(0.8, 0.9, 1, 1.2, 1.3)$

1 In particular, for each of the points there are 4! ways—due to index permutations—of calculating the same tangent.

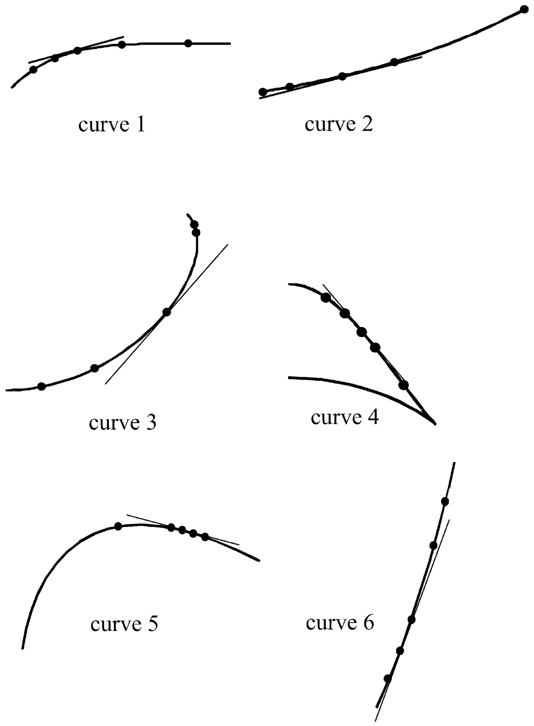


Fig. 4. Test curves: five points were extracted from the curves above. The exact tangent is drawn, and the equations are given in the Table 1.

Table 2
Experimental results: refer to Table 1 for the curve associated with each number

Curve	$\varepsilon_{\text{Fmill}}$	$\varepsilon_{\text{Bessel}}$	$\varepsilon_{\text{Akima}}$	$\varepsilon_{\text{circle}}$	$\varepsilon_{\text{conic}}$
1	0.040533	0.014225	0.013154	0.014225	0.002506
2	0.002135	0.001854	0.001994	0.001854	0.000000
3	0.047800	0.128474	0.088137	0.128474	0.008326
4	0.012574	0.005917	0.009246	0.005917	0.001150
5	0.000492	0.001228	0.000860	0.001229	0.000000
6	0.017061	0.001753	0.007654	0.001754	0.000000

The angle ε in radians measures the difference between the exact tangent and the estimate.

the interpolating circle as well as Akima's tangent are shown. All of the considered tangent estimators yield reasonable results for convex data, while a remarkably better performance has been observed for the conic based method. This method is thus suggested in the important application area of convexity preserving interpolation.

The number of computations in the conic based method is roughly ten times less than computing the conic directly via a determinant as discussed in Farin (1999) in Eq. (4.26). Numerically, the results are equally as good as computing the conic directly.

Numerical problems in the degenerate cases mentioned in Section 2.2.2 are avoided by upfront testing the data for the configurations of Fig. 3 with respect to a prescribed tolerance. If with respect to the given tolerance such a degenerate configuration is detected we propose FMILL's method for determining the desired tangent in this case.

3. Approximation order

We now study the approximation order of the tangent estimators considered in Section 2.3 in order to explain their different behavior found experimentally. To this end we consider a planar curve c, which in the neighborhood of a

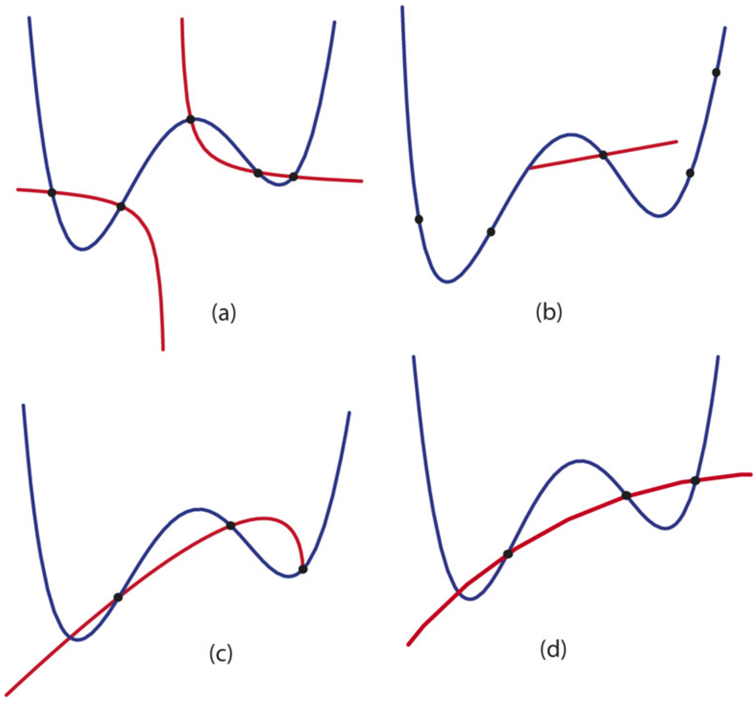


Fig. 5. Non-convex situation: erroneous tangents are obtained by the conic method (a) as tangent to the interpolating hyperbola at the middle point, by Akima's method (b), by Bessel's method (c) as tangent to the interpolating parabola at the middle point, and by the circle based method (d) as tangent to the interpolating circle at the middle point.

finite, regular point may be written in homogeneous form as

$$c\colon\ \bar{\mathbf{y}}(t) = \begin{pmatrix} 1 \\ t \\ f(t) \end{pmatrix}, \tag{7}$$

where f is a sufficiently smooth function. We now take five points[2]

$$\bar{\mathbf{p}}_i = \bar{\mathbf{y}}(t_i), \quad i = 1, \ldots, 5, \tag{8}$$

from c. Without loss of generality we assume

$$t_3 = 0 \tag{9}$$

as well as

$$f(0) = f'(0) = 0. \tag{10}$$

3.1. Theoretical results

Under the above assumptions the asymptotic Taylor expansion of $\bar{\mathbf{y}}(t)$ then reads

$$\bar{\mathbf{y}}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sum_{k=2}^{K} \frac{t^k}{k!} \begin{pmatrix} 0 \\ 0 \\ f_k \end{pmatrix} + \mathrm{O}\big(t^{K+1}\big), \tag{11}$$

[2] The parameter values are needed for the following theoretical analysis, but are not used for carrying out the practical algorithms presented earlier.

where $f_k = \frac{d^k f}{dt^k}|_{t=0}$. The exact tangent of the curve c in $\bar{\mathbf{p}}_3 = \bar{\mathbf{y}}(0)$, in its homogeneous line coordinates, is a non-zero multiple of

$$\bar{\mathbf{y}}(0) \wedge \bar{\mathbf{y}}^{(1)}(0) = (0, 0, 1)^T, \tag{12}$$

where $\bar{\mathbf{y}}^{(1)}(0) = \frac{d\bar{\mathbf{y}}}{dt}|_{t=0}$. This is due to the following relation between a tangent vector $\mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2$ of a curve in a curve's point $\mathbf{p} \in \mathbb{E}^2$ in the inhomogeneous (affine) setting on the one hand and the homogeneous line coordinates $\bar{\mathbf{L}} = (L_0, L_1, L_2)^T$ of this tangent on the other hand:

$$\bar{\mathbf{L}} = \begin{pmatrix} \det(\mathbf{v}, \mathbf{p}) \\ v_2 \\ -v_1 \end{pmatrix}. \tag{13}$$

In order to obtain the approximation orders of the different tangent estimators we now determine the Taylor expansion of the calculated tangent in its homogeneous line coordinates for each of the considered methods, and, in the asymptotic situation[3] $t_i \to 0$, $(i \neq 3)$, we compare the result to the exact tangent's representation (12). The results of the following sections have been obtained with the help of the computer algebra system MAPLE. We use the notation

$$\bar{\mathbf{t}} = (0, 0, 1)^T = \bar{\mathbf{y}}(0) \wedge \bar{\mathbf{y}}^{(1)}(0), \qquad \bar{\mathbf{n}} = (0, 1, 0)^T.$$

3.1.1. Conic method

In order to obtain the Taylor expansion of the tangent $\bar{\mathbf{L}}_{33}$ calculated by the conic method (6) we substitute the Taylor expansion (11) for $\bar{\mathbf{y}}(t_i)$, $i = 1, 2, 4, 5$ into (6).

By introducing $\alpha_{ij} := \alpha_{ij}(t_1, t_2, t_4, t_5)$, $i = 1, \ldots, 5$, $j = 1, \ldots, i$, standing for symmetric polynomials[4] in the variables t_1, t_2, t_4, t_5, containing exclusively terms of degree i, the first few terms in the Taylor expansion of $\bar{\mathbf{L}}_{33}$ are found to be the following:[5,6]

Degree 0:

$$-\frac{1}{8} f_2^3 \cdot \bar{\mathbf{t}}$$

Degree 1:

$$-\frac{1}{12} f_2^2 f_3 \alpha_{11} \cdot \bar{\mathbf{t}}$$

Degree 2:

$$-\frac{1}{288} f_2 \{3 f_2 f_4 \alpha_{21} + 4 f_3^2 \alpha_{22}\} \cdot \bar{\mathbf{t}}$$

Degree 3:

$$-\frac{1}{4320} \{30 f_2 f_3 f_4 \alpha_{31} + 9 f_2^2 f_5 \alpha_{32} + 20 f_3^3 \alpha_{33}\} \cdot \bar{\mathbf{t}}$$

Degree 4:

$$-\frac{1}{17280} \{6 f_2^2 f_6 \alpha_{41} + 20 f_3^2 f_4 \alpha_{42} + 15 f_2 f_4^2 \alpha_{43} + 24 f_2 f_3 f_5 \alpha_{44}\} \cdot \bar{\mathbf{t}}$$
$$- \frac{1}{8640} f_2 \{9 f_2^2 f_5 - 45 f_2 f_3 f_4 + 40 f_3^3\} \cdot t_1 t_2 t_4 t_5 \cdot \bar{\mathbf{n}}$$

[3] This follows with $h \to 0$, where $h = \max_i |t_i - t_3|$.

[4] The exact expressions for the α_{ij} may be obtained by contacting the authors or on the website http://www.univ-valenciennes.fr/lamav/ under prepublications.

[5] By considering $t_3 = 0$.

[6] The same result is of course obtained by calculating the tangent from the implicit equation of the conic through the five given points after substitution of the Taylor expansions for $f(t_i)$, $i \in \{1, 2, 4, 5\}$.

Degree 5:

$$-\frac{1}{120960}\left\{28f_2f_3f_6\alpha_{51}+35f_3f_4^2\alpha_{52}+42f_2f_4f_5\alpha_{53}+56f_3^2f_5\alpha_{54}+6f_2^2f_7\alpha_{55}\right\}\cdot\bar{\mathbf{t}}$$
$$-\frac{1}{51840}\left\{9f_2^3f_6-45f_2^2f_4^2+30f_2f_3^2f_4-18f_2^2f_3f_5+40f_3^4\right\}\cdot t_1t_2t_4t_5\cdot\left(\sum_{i=1}^{5}t_i\right)\cdot\bar{\mathbf{n}}$$

For $h := \max_{k\in\{1,2,4,5\}}|t_k|$ the asymptotic behavior of the $\alpha_{ij}=\alpha_{ij}(t_1,t_2,t_4,t_5)$, may be expressed as follows:

$$\alpha_{i,j}(t_1,t_2,t_4,t_5)=\mathrm{O}(h^i)\quad\text{for } h\to 0.$$

We thus can formulate

Theorem 2. *Let*

$$c\colon\ \bar{\mathbf{y}}(t)=\begin{pmatrix}1\\ t\\ f(t)\end{pmatrix}$$

be a sufficiently smooth planar curve in its homogeneous parameter form, where $f(0)=f'(0)=0$*, and let* $\bar{\mathbf{t}}=(0,0,1)^T$ *respectively* $\bar{\mathbf{n}}=(0,1,0)^T$ *denote the tangent respectively the normal of c in* $\bar{\mathbf{y}}(0)$ *in homogeneous line coordinates. Let then* $\bar{\mathbf{p}}_i=\bar{\mathbf{y}}(t_i)$, $i=1,\dots,5$ *be five points on c where* $t_3=0$*, thus* $\bar{\mathbf{p}}_3=\bar{\mathbf{y}}(0)$*, and let* $\bar{\mathbf{L}}_{33}$ *be the line calculated from these five points by algorithm* (6).

If $\bar{\mathbf{p}}_3$ *is not an inflection point of c* ($f_2\neq 0$) *it then holds* (*for* $h\to 0$):

$$\bar{\mathbf{L}}_{33}=\mathrm{O}(1)\cdot\bar{\mathbf{t}}+\mathrm{O}(h^4)\cdot\bar{\mathbf{n}}.$$

If $\bar{\mathbf{p}}_3$ *is an inflection point of c* ($f_2=0,\ f_3\neq 0$) *then we have* (*for* $h\to 0$):

$$\bar{\mathbf{L}}_{33}=\mathrm{O}(h^3)\cdot\bar{\mathbf{t}}+\mathrm{O}(h^5)\cdot\bar{\mathbf{n}}.$$

In convex settings the tangent estimation algorithm (6) thus reproduces the desired tangent with approximation order *four*, whereas in configurations with inflection points the approximation order reduces to *two*.

3.1.2. FMILL's method

According to (1) the estimated tangent vector $\mathbf{v}\in\mathbb{R}^2$ of the curve c in $\mathbf{p}_3$ in its inhomogeneous representation $\mathbf{y}(t)=(t,f(t))^T$ is given by $\mathbf{v}=\mathbf{p}_4-\mathbf{p}_2$. Applying (13) this yields the homogeneous line coordinates of the desired tangent as

$$\bar{\mathbf{L}}=\begin{pmatrix}0\\ f(t_4)-f(t_2)\\ t_2-t_4\end{pmatrix}.\tag{14}$$

After division by the factor t_2-t_4 the Taylor polynomial of $\bar{\mathbf{L}}$ thus reads:

$$\bar{\mathbf{t}}-\left(\sum_{k=2}^{m}\left(\sum_{j=0}^{k-1}t_4^jt_2^{k-1-j}\right)\frac{f_k}{k!}\right)\cdot\bar{\mathbf{n}}.$$

Since for $h := \max_{l\in\{2,4\}}|t_l|$ we have

$$\sum_{j=0}^{k-1}t_4^jt_2^{k-1-j}=\mathrm{O}(h^{k-1})\quad\text{for } h\to 0,$$

we thus obtain

Theorem 3. *Let*

$$c\colon\ \bar{\mathbf{y}}(t)=\begin{pmatrix}1\\ t\\ f(t)\end{pmatrix}$$

be a sufficiently smooth planar curve in its homogeneous parameter form, where $f(0)=f'(0)=0$, *and let* $\bar{\mathbf{t}}=(0,0,1)^T$ *respectively* $\bar{\mathbf{n}}=(0,1,0)^T$ *denote the tangent respectively the normal of* c *in* $\bar{\mathbf{y}}(0)$ *in homogeneous line coordinates. Let then* $\bar{\mathbf{p}}_i=\bar{\mathbf{y}}(t_i)$, $i=2,3,4$ *be three points on* c *where* $t_3=0$, *thus* $\bar{\mathbf{p}}_3=\bar{\mathbf{y}}(0)$, *and let* $\bar{\mathbf{L}}$ *be the line* (14).

If $\bar{\mathbf{p}}_3$ *is not an inflection point of* c $(f_2\neq 0)$ *it then holds* (*for* $h\to 0$):

$$\bar{\mathbf{L}}=\mathrm{O}(1)\cdot\bar{\mathbf{t}}+\mathrm{O}(h)\cdot\bar{\mathbf{n}}.$$

If $\bar{\mathbf{p}}_3$ *is an inflection point of* c $(f_2=0,\ f_3\neq 0)$ *then we have* (*for* $h\to 0$):

$$\bar{\mathbf{L}}=\mathrm{O}(1)\cdot\bar{\mathbf{t}}+\mathrm{O}\big(h^2\big)\cdot\bar{\mathbf{n}}.$$

In the case of convex point configurations FMILL's method thus approximates the desired tangent with approximation order *one*, whereas in an inflection point the approximation order becomes *two*.

3.1.3. Bessel's method

According to (2) Bessel's method obtains the desired tangent as tangent to the interpolating parabola $\mathbf{x}(\tau)\subset\mathbb{E}^2$ to the points $\mathbf{x}(\tau_i)=\mathbf{y}(t_i)=\mathbf{p}_i$, $i=2,3,4$. The parameter values τ_2,τ_3,τ_4 have to be estimated, e.g., by chord length parametrization. Whatever parameter estimator is used we may assume without loss of generality $\tau_3=0$. The tangent vector $\mathbf{v}\in\mathbb{R}^2$ is thus calculated to be

$$\mathbf{v}=\tau_4^2\big(\mathbf{y}(t_2)-\mathbf{y}(0)\big)-\tau_2^2\big(\mathbf{y}(t_4)-\mathbf{y}(0)\big).$$

Applying (13) thus yields the following homogeneous line coordinates for the desired tangent $\bar{\mathbf{L}}$.

$$\bar{\mathbf{L}}=\begin{pmatrix}0\\ \tau_4^2 f(t_2)-\tau_2^2 f(t_4)\\ -\tau_4^2 t_2+\tau_2^2 t_4\end{pmatrix}. \tag{15}$$

The Taylor polynomial of $\bar{\mathbf{L}}$ thus reads

$$\big(t_4\tau_2^2-t_2\tau_4^2\big)\cdot\bar{\mathbf{t}}-\left\{\sum_{k=2}^{m}\left[\big(t_4^k\tau_2^2-t_2^k\tau_4^2\big)\frac{f_k}{k!}\right]\right\}\cdot\bar{\mathbf{n}}.$$

Since for $h:=\max_{l\in\{2,4\}}|t_l|$ we have

$$t_4^k\tau_2^2-t_2^k\tau_4^2=\mathrm{O}\big(h^k\big)\quad\text{for } h\to 0,$$

we thus obtain

Theorem 4. *Let*

$$c\colon\ \bar{\mathbf{y}}(t)=\begin{pmatrix}1\\ t\\ f(t)\end{pmatrix}$$

be a sufficiently smooth planar curve in its homogeneous parameter form, where $f(0)=f'(0)=0$, *and let* $\bar{\mathbf{t}}=(0,0,1)^T$ *respectively* $\bar{\mathbf{n}}=(0,1,0)^T$ *denote the tangent respectively the normal of* c *in* $\bar{\mathbf{y}}(0)$ *in homogeneous line coordinates. Let then* $\bar{\mathbf{p}}_i=\bar{\mathbf{y}}(t_i)$, $i=2,3,4$ *be three points on* c *where* $t_3=0$, *thus* $\bar{\mathbf{p}}_3=\bar{\mathbf{y}}(0)$, *and let* $\bar{\mathbf{L}}$ *be the line* (2), *obtained as tangent to the interpolating parabola* $\bar{\mathbf{x}}(\tau)$ *to the three points* $\bar{\mathbf{p}}_2$, $\bar{\mathbf{p}}_3$, $\bar{\mathbf{p}}_4$.

If $\bar{\mathbf{p}}_3$ *is not an inflection point of* c $(f_2\neq 0)$ *it then holds* (*for* $h\to 0$):

$$\bar{\mathbf{L}}=\mathrm{O}(h)\cdot\bar{\mathbf{t}}+\mathrm{O}\big(h^2\big)\cdot\bar{\mathbf{n}}.$$

If $\bar{\mathbf{p}}_3$ *is an inflection point of* c $(f_2=0,\ f_3\neq 0)$ *then we have* (*for* $h\to 0$):

$$\bar{\mathbf{L}}=\mathrm{O}(h)\cdot\bar{\mathbf{t}}+\mathrm{O}\big(h^3\big)\cdot\bar{\mathbf{n}}.$$

The approximation order of Bessel's method in general is thus one. Only in inflection points or in the case where the parameters of c and those of the interpolating parabola coincide, i.e., $t_i = \tau_i$, $i = 2, 3, 4$, also the second order term vanishes, and thus the approximation order increases to two. Since in real world applications we usually do not know the underlying curve c, and much less its parametrization, we thus have to consider Bessel's method as a first order method in convex settings with the possibility of influencing the result by choosing an appropriate parametrization. For example, for chord length parametrization, i.e.,

$$\tau_2 = -\|\mathbf{p}_3 - \mathbf{p}_2\|_2, \qquad \tau_3 = 0, \qquad \tau_4 = \|\mathbf{p}_4 - \mathbf{p}_3\|_2, \tag{16}$$

the tangent (15) becomes

$$\bar{\mathbf{L}} = \begin{pmatrix} 0 \\ (t_4^2 + f(t_4)^2) f(t_2) - (t_2^2 + f(t_2)^2) f(t_4) \\ -(t_4^2 + f(t_4)^2) t_2 + (t_2^2 + f(t_2)^2) t_4 \end{pmatrix}.$$

The first few terms of $\bar{\mathbf{L}}$'s Taylor expansion thus read:

Degree 0:

$$-\bar{\mathbf{t}}$$

Degree 1:

$$\underline{\qquad\qquad}$$

Degree 2:

$$-\frac{1}{4} f_2^2 \left(t_2^2 + t_2 t_4 + t_4^2\right) \cdot \bar{\mathbf{t}} - \frac{1}{6} f_3 t_2 t_4 \cdot \bar{\mathbf{n}}$$

Degree 3:

$$-\frac{1}{6} f_2 f_3 \left(t_2^3 + t_2^2 t_4 + t_2 t_4^2 + t_4^3\right) \cdot \bar{\mathbf{t}} + \frac{1}{24}\left(2 f_2^3 - f_4\right) t_2 t_4 (t_2 + t_4) \cdot \bar{\mathbf{n}}$$

Degree 4:

$$-\frac{1}{72}\left(2 f_3^2 + 3 f_2 f_4\right) \cdot \left(t_2^4 + t_2^3 t_4 + t_2^2 t_4^2 + t_2 t_4^3 + t_4^4\right) \cdot \bar{\mathbf{t}}$$
$$+ \frac{1}{120}\left(\left(10 f_2^2 f_3 - f_5\right) t_2 t_4 \left(t_2^2 + t_2 t_4 + t_4^2\right) + 5 f_2^2 f_3 t_2^2 t_4^2\right) \cdot \bar{\mathbf{n}}$$

We thus obtain the following corollary to Theorem 4.

Corollary 1. *For the situation of Theorem* 4 *we suppose the interpolating parabola* $\bar{\mathbf{x}}(\tau)$ *to the three points* $\bar{\mathbf{p}}_2$, $\bar{\mathbf{p}}_3$, $\bar{\mathbf{p}}_4$ *to be parametrised by the chord length method* (*see* (16)).

Independently of the fact whether $\bar{\mathbf{p}}_3$ *is an inflection point of* c ($f_2 = 0,\ f_3 \neq 0$) *or not* ($f_2 \neq 0$) *it then holds* (*for* $h \to 0$):

$$\bar{\mathbf{L}} = \mathrm{O}(1) \cdot \bar{\mathbf{t}} + \mathrm{O}\left(h^2\right) \cdot \bar{\mathbf{n}}.$$

In the case of chord length parametrization Bessel's method is thus of approximation order *two*.

3.1.4. Akima's method

In the case of Akima's method, according to (13) and (3), and under the assumptions (9), (10), we obtain the estimated tangent in its homogeneous line coordinates as

$$\bar{\mathbf{L}} = \begin{pmatrix} 0 \\ \sqrt{\beta(t_4, t_5)}\tau_4 f(t_2) + \sqrt{\gamma(t_1, t_2)}\tau_2 f(t_4) \\ -\sqrt{\beta(t_4, t_5)}\tau_4 t_2 - \sqrt{\gamma(t_1, t_2)}\tau_2 t_4 \end{pmatrix}, \tag{17}$$

where

$$\beta(t_4,t_5)=\frac{(t_4\tau_5-t_5\tau_4)^2+(f(t_4)\tau_5-f(t_5)\tau_4)^2}{(\tau_5-\tau_4)^2\tau_4^2},$$
$$\gamma(t_1,t_2)=\frac{(t_1\tau_2-t_2\tau_1)^2+(f(t_1)\tau_2-f(t_2)\tau_1)^2}{(\tau_2-\tau_1)^2\tau_2^2}.$$

Since analytically $\beta(0,0)=\gamma(0,0)=0$, $\beta(t_4,t_5)$ and $\gamma(t_1,t_2)$ are not differentiable in $(0,0)$, and thus no Taylor expansion exists for $\bar{\mathbf{L}}$ in $(t_1,t_2,t_4,t_5)=(0,0,0,0)$. We thus consider a slightly perturbed tangent of the form

$$\bar{\mathbf{L}}_{\mathrm{num}}=\begin{pmatrix}0\\ \sqrt{\beta(t_4,t_5)+\epsilon_1}\tau_4 f(t_2)+\sqrt{\gamma(t_1,t_2)+\epsilon_2}\tau_2 f(t_4)\\ -\sqrt{\beta(t_4,t_5)+\epsilon_1}\tau_4 t_2-\sqrt{\gamma(t_1,t_2)+\epsilon_2}\tau_2 t_4\end{pmatrix}, \tag{18}$$

for small positive values $\epsilon_1,\epsilon_2\in\mathbb{R}^+$. By calculating the Taylor expansion of $\bar{\mathbf{L}}_{\mathrm{num}}$ we obtain the following first few terms.

Degree 1:

$$\delta_{1,1}(t_1,t_2,t_4,t_5)\cdot\bar{\mathbf{t}},$$

where $\delta_{1,1}(t_1,t_2,t_4,t_5)=-(\sqrt{\epsilon_1}\tau_4t_2+\sqrt{\epsilon_2}\tau_2t_4)$.

Degree 2:

$$\delta_{2,2}(t_1,t_2,t_4,t_5)\cdot\bar{\mathbf{n}},$$

where $\delta_{2,2}(t_1,t_2,t_4,t_5)=\frac{1}{2}f_2(\sqrt{\epsilon_1}\tau_4t_2^2+\sqrt{\epsilon_2}\tau_2t_4^2)$.

Degree 3:

$$\delta_{3,1}(t_1,t_2,t_4,t_5)\cdot\bar{\mathbf{t}}+\delta_{3,2}(t_1,t_2,t_4,t_5)\cdot\bar{\mathbf{n}},$$

where

$$\begin{aligned}\delta_{3,1}(t_1,t_2,t_4,t_5)=&-\frac{\tau_4}{2(\tau_5-\tau_4)^2\sqrt{\epsilon_1}}t_2t_5^2+\frac{\tau_5}{(\tau_5-\tau_4)^2\sqrt{\epsilon_1}}t_2t_4t_5\\&-\frac{\tau_1^2}{2\tau_2(\tau_1-\tau_2)^2\sqrt{\epsilon_2}}t_2^2t_4+\frac{\tau_1}{(\tau_1-\tau_2)^2\sqrt{\epsilon_2}}t_1t_2t_4\\&-\frac{\tau_2}{2(\tau_2-\tau_1)^2\sqrt{\epsilon_2}}t_1^2t_4-\frac{\tau_5^2}{2\tau_4(\tau_5-\tau_4)^2\sqrt{\epsilon_1}}t_2t_4^2,\\ \delta_{3,2}(t_1,t_2,t_4,t_5)=&\frac{1}{6}f_3\big(\sqrt{\epsilon_1}\tau_4t_2^3+\sqrt{\epsilon_2}\tau_2t_4^3\big).\end{aligned}$$

We now consider a sequence of such tangents $\bar{\mathbf{L}}_{\mathrm{num}}$ for $\epsilon_k\to 0$, $k=1,2$, where $\lim_{\epsilon_k\to 0}\bar{\mathbf{L}}_{\mathrm{num}}=\bar{\mathbf{L}}$.

Since for $h:=\max\{\max_{l\in\{1,2,4,5\}}|t_l|,\max_{k\in\{1,2\}}\epsilon_k\}$ we have for $i\in\{1,2,3\}$, $j\in\{1,2\}$:

$$\delta_{i,j}(t_1,t_2,t_4,t_5,\epsilon_1,\epsilon_2)=\begin{cases}\mathrm{O}(h^{i+\frac{1}{2}}) & \text{if } i-j\leqslant 1,\\ \mathrm{O}(h^{i-\frac{1}{2}}) & \text{if } i-j>1,\end{cases}\quad \text{for } h\to 0,$$

we can thus formulate

Theorem 5. *Let*

$$c\colon\ \bar{\mathbf{y}}(t)=\begin{pmatrix}1\\ t\\ f(t)\end{pmatrix}$$

be a sufficiently smooth planar curve in its homogeneous parameter form, where $f(0) = f'(0) = 0$*, and let* $\bar{\mathbf{t}} = (0,0,1)^T$ *respectively* $\bar{\mathbf{n}} = (0,1,0)^T$ *denote the tangent respectively the normal of c in* $\bar{\mathbf{y}}(0)$ *in homogeneous line coordinates. Let then* $\bar{\mathbf{p}}_i = \bar{\mathbf{y}}(t_i)$*,* $i = 1,\dots,5$*, be five points on c where* $t_3 = 0$*, thus* $\bar{\mathbf{p}}_3 = \bar{\mathbf{y}}(0)$*, and let* $\bar{\mathbf{L}}_{\text{num}}$ *be the line calculated from these five points according to* (18)*. If* $\bar{\mathbf{p}}_3$ *is not an inflection point of c* ($f_2 \neq 0$) *it then holds* (*for* $h \to 0$):

$$\bar{\mathbf{L}}_{\text{num}} = \mathrm{O}\big(h^{\frac{3}{2}}\big)\cdot\bar{\mathbf{t}} + \mathrm{O}\big(h^{\frac{5}{2}}\big)\cdot\bar{\mathbf{n}}.$$

If $\bar{\mathbf{p}}_3$ *is an inflection point of c* ($f_2 = 0,\ f_3 \neq 0$) *then we have* (*for* $h \to 0$):

$$\bar{\mathbf{L}}_{\text{num}} = \mathrm{O}\big(h^{\frac{3}{2}}\big)\cdot\bar{\mathbf{t}} + \mathrm{O}\big(h^{\frac{7}{2}}\big)\cdot\bar{\mathbf{n}}.$$

We thus conclude that Akima's method is a *first* order method for convex point configurations, its approximation order increases to *two* in an inflection point.

3.1.5. Circle method

In the case of the circle method according to (4) we obtain the estimated tangent in its homogeneous line coordinates as

$$\bar{\mathbf{L}} = \begin{pmatrix} 0 \\ t_4^2 f(t_2) - t_2^2 f(t_4) + f(t_2) f(t_4)^2 - f(t_2)^2 f(t_4) \\ t_2^2 t_4 - t_4^2 t_2 + t_4 f(t_2)^2 - t_2 f(t_4)^2 \end{pmatrix}. \tag{19}$$

The first few terms of its Taylor expansion are calculated to be:

Degree 3:

$$\big(t_2^2 t_4 - t_2 t_4^2\big)\cdot\bar{\mathbf{t}}$$

Degree 4:

———

Degree 5:

$$\frac{1}{4} f_2\big(t_2^4 t_4 - t_2 t_4^4\big)\cdot\bar{\mathbf{t}} + \frac{1}{6} f_3\big(t_2^3 t_4^2 - t_2^2 t_4^3\big)\cdot\bar{\mathbf{n}}$$

Degree 6:

$$\frac{1}{6} f_2 f_3\big(t_2^5 t_4 - t_2 t_4^5\big)\cdot\bar{\mathbf{t}} + \frac{1}{24}\big(f_4 - 3f_2^3\big)\big(t_2^4 t_4^2 - t_2^2 t_4^4\big)\cdot\bar{\mathbf{n}}$$

Since for $h := \max_{l\in\{2,4\}} |t_l|$ we have

$$t_2^i t_4^j - t_2^j t_4^i = \mathrm{O}\big(h^{i+j}\big) \quad \text{for } h \to 0,$$

we can thus formulate:

Theorem 6. *Let*

$$c : \bar{\mathbf{y}}(t) = \begin{pmatrix} 1 \\ t \\ f(t) \end{pmatrix}$$

be a sufficiently smooth planar curve in its homogeneous parameter form, where $f(0) = f'(0) = 0$*, and let* $\bar{\mathbf{t}} = (0,0,1)^T$ *respectively* $\bar{\mathbf{n}} = (0,1,0)^T$ *denote the tangent respectively the normal of c in* $\bar{\mathbf{y}}(0)$ *in homogeneous line coordinates. Let then* $\bar{\mathbf{p}}_i = \bar{\mathbf{y}}(t_i)$*,* $i = 2,3,4$*, be three points on c where* $t_3 = 0$*, thus* $\bar{\mathbf{p}}_3 = \bar{\mathbf{y}}(0)$*, and let* $\bar{\mathbf{L}}$ *be the line* (19)*, obtained as tangent to the interpolating circle to the three points* $\bar{\mathbf{p}}_2$*,* $\bar{\mathbf{p}}_3$*,* $\bar{\mathbf{p}}_4$*.*

Independently of the fact whether $\bar{\mathbf{p}}_3$ *is an inflection point of* c $(f_2 = 0,\ f_3 \neq 0)$ *or not* $(f_2 \neq 0)$ *it then holds* (*for* $h \to 0$):

$$\bar{\mathbf{L}} = \mathrm{O}\big(h^3\big) \cdot \bar{\mathbf{t}} + \mathrm{O}\big(h^5\big) \cdot \bar{\mathbf{n}}.$$

The circle method is thus a method of approximation order *two* whether or not we have a configuration with inflection points.

3.2. Numerical results

In this section we illustrate the above theoretical results by numerical examples. To this end we consider a given curve (7) under the assumptions (8)–(10). We choose the sequence

$$h_k = 2^{-k}, \quad k = 0, 1, 2, \ldots,$$

and randomly generate[7] parameter values $t_i(h_k)$, $i = 1, 2, 4, 5$, where $t_i(h_k) \to 0$ for $k \to \infty$. For each value of k we then apply the different tangent estimators to the points $(t_i, f(t_i))$, thus obtaining the respective estimated tangent in the point $(t_3, f(t_3))$. We then calculate the angle $\epsilon_k = \epsilon(h_k)$ in radians between the calculated tangent and the exact tangent. Since

$$\frac{h_k}{h_{k-1}} = \frac{1}{2}$$

a method of approximation order n satisfies

$$\frac{\epsilon_k}{\epsilon_{k-1}} = \mathrm{O}\left(\frac{1}{2^n}\right).$$

In order to enhance readability of the numerical results we display the dual logarithms of h_k and ϵ_k, where

$$\log_2 h_k = -k = \log_2 h_{k-1} - 1,$$
$$\log_2 \epsilon_k \approx \log_2 \epsilon_{k-1} - n.$$

For every example we display the graph of the curve with its exact tangent, a comparative table showing the values $|\log_2 h_k|$ and $|\log_2 \epsilon_k|$ for every method as well as a graphical illustration thereof.

Fig. 6 illustrates the situation for the curve $(t, \exp(t) - t - 1)$, where the considered point $(0, 0)$ is not an inflection point. For Bessel's and Akima's methods we chose chord length parametrization in this example. The numerical results confirm the theoretical ones: the approximation order of the conic method is 4, Bessel's method with chord length parametrization and the circle based method have approximation order 2, and FMILL and Akima's methods have approximation order 1.

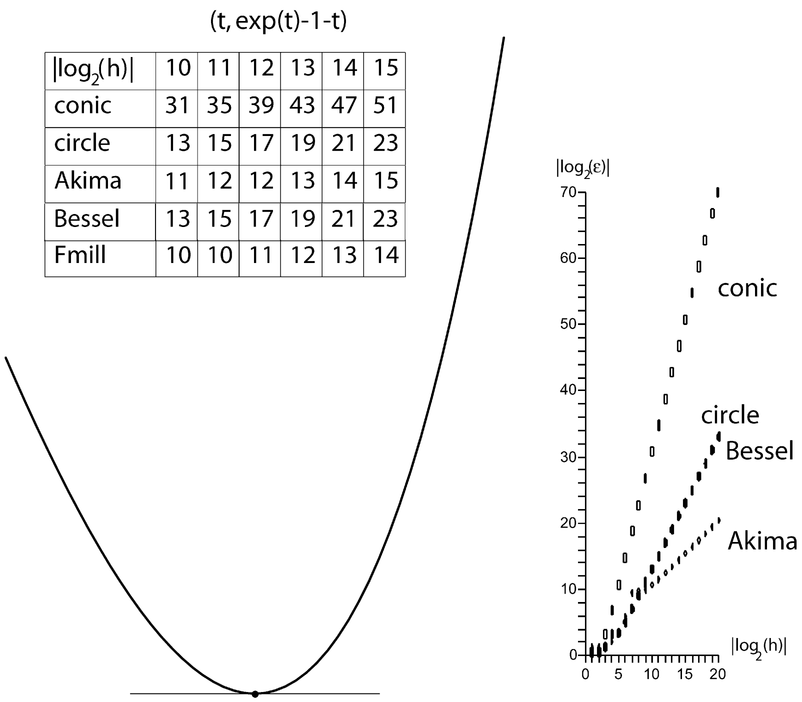
Fig. 7 illustrates the situation for the curve $(t, t^4 + t^3)$, where the considered point $(0, 0)$ is an inflection point. For Bessel's and Akima's methods we chose uniform parametrization in this example. The numerical results confirm the theoretical ones: the approximation order of all the methods is 2.

Fig. 8 illustrates the situation for the curve $(t, \frac{t^2}{1+t^2})$, where the considered point $(0, 0)$ is not an inflection point. For Bessel's and Akima's methods we chose uniform parametrization in this example. The numerical results confirm the theoretical ones: the approximation order of the conic method is 4, the circle based method has approximation order 2, and Bessel's, Akima's, and FMILL's methods have approximation order 1.

4. Conclusions

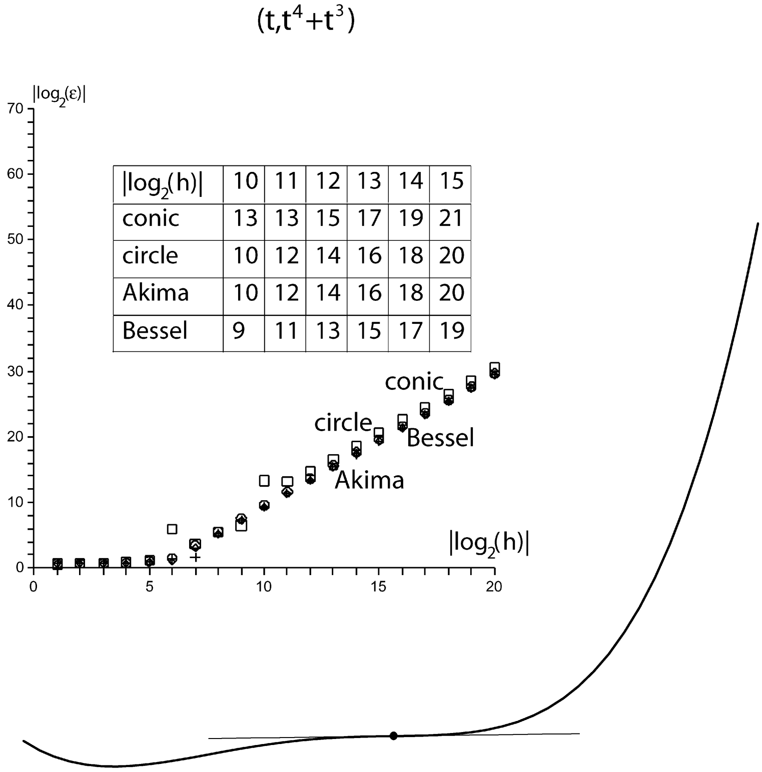
We have presented a method of tangent estimation for planar, convex data which is based on the classical theorem of Pascal and achieves conic precision. Numerous examples have illustrated a remarkably better performance of the presented "conic" method with respect to classical tangent estimation schemes such as FMILL, Bessel, Akima's, and the circle based method.

[7] For example, by taking $t_i(h_k) = \sum_{j=1}^{d} \gamma_{ij} h_k^j / j!$ with randomly chosen coefficients γ_{ij} and $d \geqslant 1$.



$\lvert\log_2(h)\rvert$	10	11	12	13	14	15
conic	31	35	39	43	47	51
circle	13	15	17	19	21	23
Akima	11	12	12	13	14	15
Bessel	13	15	17	19	21	23
Fmill	10	10	11	12	13	14

Fig. 6. Comparison of the tangent estimators for points taken from the curve $(t, \exp(t) - t - 1)$. The values of $|\log_2(\epsilon)|$ for the different methods are shown in the table and partially illustrated in the graph.



$\lvert\log_2(h)\rvert$	10	11	12	13	14	15
conic	13	13	15	17	19	21
circle	10	12	14	16	18	20
Akima	10	12	14	16	18	20
Bessel	9	11	13	15	17	19

Fig. 7. Comparison of the tangent estimators for points taken from the curve $(t, t^4 + t^3)$. The values of $|\log_2(\epsilon)|$ for the different methods are shown in the table and illustrated in the graph.

$(t, \frac{t^2}{1+t^2})$

$\lvert \log_2(h) \rvert$	10	11	12	13	14	15
conic	26	30	34	38	42	46
circle	15	17	19	21	23	25
Akima	8	9	10	11	12	13
Bessel	11	12	13	14	15	16
Fmill	11	12	13	14	15	16

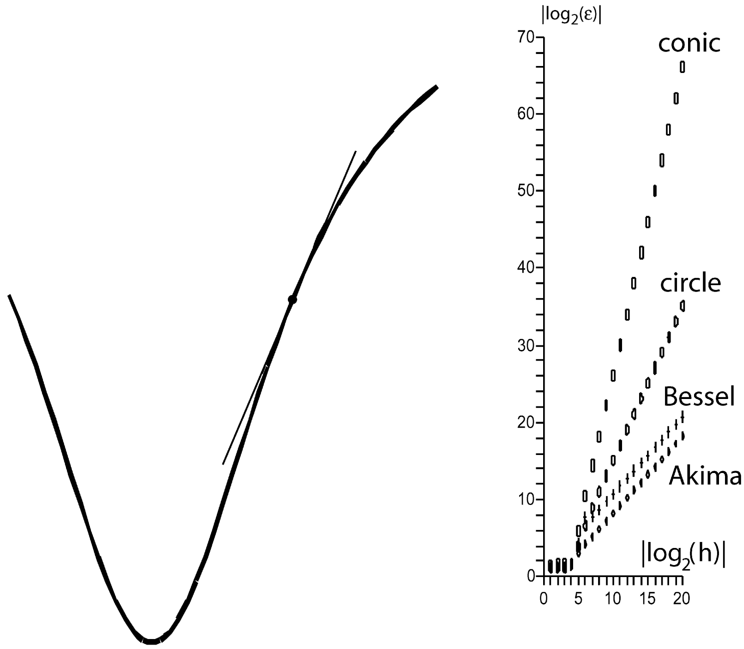


Fig. 8. Comparison of the tangent estimators for points taken from the curve $(t, \frac{t^2}{1+t^2})$. The values of $|\log_2(\epsilon)|$ for the different methods are shown in the table and partially illustrated in the graph.

A thorough theoretical study has substantiated the results found experimentally: in convex settings the presented conic based method has proven to be of approximation order four whereas the circle based one has shown to be of approximation order two, and FMILL's, Bessel's, and Akima's methods in general have approximation order one. In inflection points all methods have approximation order two.

In different settings, Schaback (1993), Liming (1944), and Pavlidis (1983) have presented similar methods for constructing tangents with conic precision, but neither comparisons to other existing methods nor a theoretical study of the tangent estimators approximation order has been given. The presented study thus gives a theoretical justification for using a conic based tangent estimator as well as an efficient and easy to implement scheme for obtaining tangent estimates in planar, convex data points.

Acknowledgements

The authors would like to thank the Laboratoire MACS (now part of LAMAV), of the University of Valenciennes for providing support which enabled G. Farin and D. Hansford to visit the CAGD team. They also wish to acknowledge the very useful comments of the unknown referees that helped to significantly improve the section on the approximation order of the conic based method. Special thanks go to our colleague F. Ali-Mehmeti for his thorough reading of the manuscript and his useful suggestions regarding the approximation order of Akima's method.

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