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The neutral case for the min-max triangulation

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Abstract. Choosing the best triangulation of a point set is a question that has been debated for many years. Two of the most well known choices are the min-max criterion and the max-min criterion. The max-min triangulation criterion has received the most attention over the years because efficient algorithms have been developed for determining this triangulation. The ability to construct such efficient algorithms has been shown to be a result of the geometry of the *neutral set* for the max-min criterion. A point from the neutral set is formed from the special instance when the criterion is satisfied by more than one triangulation. For the max-min criterion, the neutral set is a circle. In this paper, we construct the neutral set for the min-max criterion. This construction is compared to that of the max-min triangulation and the results are analyzed in order to attain a better understanding of the nature of the min-max criterion.

Keywords. Delaunay triangulation, max-min triangulation, min-max triangulation

0. Introduction

Two of the most commonly examined triangulation criteria are the min-max and max-min. The debate over which criteria produces the best triangulation has gone on for some time [Babuska & Aziz '76, Barnhill '83, Gregory '75]. On an algorithmic level, the primary difference between the min-max and max-min criteria is the existence of a practical algorithm for finding the triangulation of an arbitrary point set. The max-min criterion, for which practical algorithms exist [Bowyer '81, Cline & Renka '84, Correc & Chapuis '87, Field '87, Watson '81], allows for a procedure called *local optimization* which satisfies the global criterion upon termination [Lawson '77]. Local optimization means that we only need to consider sets of four points that form convex quadrilaterals; these quadrilaterals are examined for the pair of triangles that should be constructed in order to satisfy the criterion. This type of algorithm is known as a local swapping algorithm. However, local swapping has been shown in general not to yield the global optimum for the min-max criterion [Piper '86]. Therefore, the purpose of this paper is to examine the local min-max criterion in order to understand why locally optimal triangles do not imply a globally optimal triangulation.

1. The neutral case

Suppose we are given the four vertices of a convex quadrilateral and wish to subdivide to create two triangles. There are two diagonals from which to choose. A decision may be made by applying some criterion such as the min-max or max-min; these criteria will not in general choose the same diagonal. A *neutral case* for a triangulation criterion occurs when either diagonal may be chosen. Suppose we are given three points and must define a fourth such that

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a convex quadrilateral is formed. All such fourth points that result in a neutral case are elements of the neutral set for the three given points. The neutral set is well known for the max-min criterion; it is the circumcircle through the three given points. An application of this result is the *local circle test* [Lawson '77]. We will examine the local min-max criterion by defining its neutral set.

2. Neutral case for the min-max

As a first step, it is necessary to make the problem specification more exact. First of all, we need to determine just how the three given vertices may be configured since we are only dealing with convex quadrilaterals. Referring to Fig. 1, let the three vertices be given in clockwise order and be labeled v_1, v_2, v_3 and the fourth, yet unknown, vertex be labeled x. Let $l_1 = \| v_1 - v_2 \|$, $l_2 = \| v_3 - v_2 \|$, $l_3 = \| x - v_3 \|$, $l_4 = \| v_1 - x \|$, and $l_5 = \| v_1 - v_3 \|$. The angle formed by $(v_1 - v_2)$ and $(v_3 - v_2)$ will be called Γ. In addition, if we radiate a line from v_2 that divides Γ, the two angles formed will be called γ and δ. Once we have found x we will give the radiating line a length, $r = \| x - v_2 \|$.

One short proposition is necessary before the problem can be fully specified.

Proposition. *For a convex quadrilateral to be constructed such that the neutral case occurs while applying the min-max criterion, two adjacent angles must be equal and they must be the maximum angles in the quadrilateral.*

Proof. Suppose the proposition is not true. In that case, one of the following situations must occur: (1) one angle of the four is the maximum or (2) two facing angles are equal and the maximum. This follows since the maximum angle cannot be less than 90°. If (1) occurs, then we must choose the diagonal that divides this angle in order to minimize the maximum angle. Hence this situation does not yield a neutral case. If (2) occurs, then we must choose the diagonal that splits the largest angles. Since the only other configuration is that in the proposition, it is proved. □

It is necessary to split the problem of finding the neutral set for the min-max criterion into two problems.

Problem 1. $\Gamma \leqslant 90°$.

From the proposition we need to choose two maximum adjacent angles. Let one be at x and have magnitude α. The other may be either at v_1 or v_3. We may choose this arbitrarily, so let it be at v_1 for now. This problem is illustrated in Fig. 1.

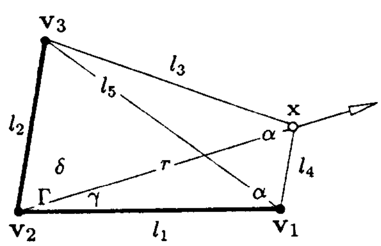

Fig. 1. The labeling used to approach the problem

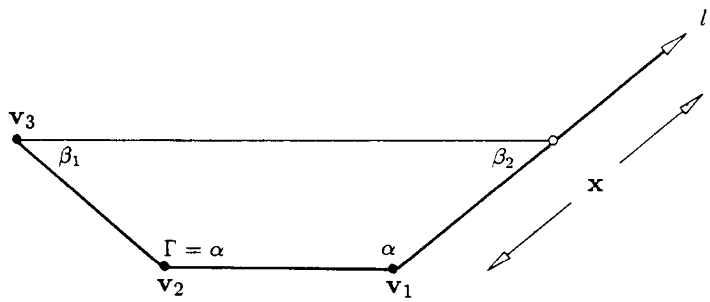

Fig. 2. A geometric description of the solution for Problem 2, case 1. The angles β_1 and β_2 cannot exceed α. $\Gamma = \alpha = 140°$ and the angle at v_1 is equal to 140°. The fourth vertex x can be chosen along a segment of l such that β_1 and β_2 are less than 140°.

Problem 2. $\Gamma > 90°$.

This problem is more involved than the former. Because Γ is being considered to be greater than 90°, there are two distinct cases in Problem 2: $\Gamma = \alpha$ and $\Gamma \neq \alpha$. Of course for each of these cases there are two positionings of the α. One instance of each case 1 and case 2 is illustrated in Figs. 2 and 3, respectively. Simple calculations reveal that case 2 can occur only if $90° < \Gamma < 120°$.

Reiterating, the problem in a general sense is the following: given three vertices v_1, v_2, and v_3 in clockwise order such that the angle at v_2 is Γ, find the loci of a fourth vertex x such that the convex quadrilateral formed results in a neutral case when the min-max criterion is applied. The solution will be presented in two parts: a solution to Problem 1 and then a solution to Problem 2. As a result, if we are given three arbitrary vertices, the appropriate solution must be applied to each of the three angles formed.

3. The solution to Problem 1

The solution to this problem simply reduces to repeated application of the law of cosines. Consequently, polar coordinates are used. First, while examining Fig. 1, we have the following

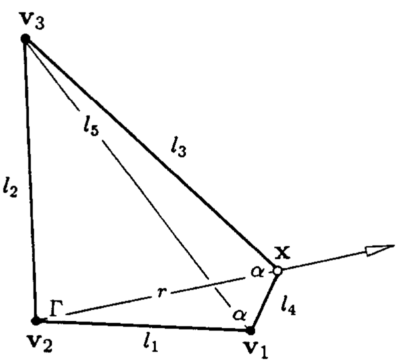

Fig. 3. The labeling for Problem 2, case 2 with $\Gamma > 90°$, $\Gamma \neq \alpha$, and the angles at v_1 and x are equal to α. A constraint of $90° < \Gamma < 120°$ is given.

relationships using the law of cosines:

$$r^2 = l_1^2 + l_4^2 - 2l_1l_4 \cos \alpha, \tag{1}$$

$$l_5^2 = l_3^2 + l_4^2 - 2l_3l_4 \cos \alpha, \tag{2}$$

$$l_3^2 = l_2^2 + r^2 - 2l_2r \cos \delta, \tag{3}$$

$$l_4^2 = l_1^2 + r^2 - 2l_1r \cos \gamma. \tag{4}$$

Equate equations (1) and (2) with their cos α term and substitute equations (3) and (4) into this. We have

$$\frac{\left(-2l_1^2 + 2l_1r \cos \gamma\right)}{l_1} = \frac{\left(-l_1^2 - l_2^2 + l_5^2 - 2r^2 + 2l_2r \cos \delta + 2l_1r \cos \gamma\right)}{\left(l_2^2 + r^2 - 2l_2r \cos \delta\right)^{1/2}} \tag{5}$$

were r is the only unknown. This equation can be rearranged, but not really simplified.

Let us discuss the application of equation (5). Consider a series of rays emanating from v_2 with an angle called γ whose values vary between zero and σ. As in Fig. 1, on each ray there will be a point x that is a distance r from v_2 such that the above expression is satisfied. Since we chose one α to be at v_1, this equation is only relevant if the angle at x and the angle at v_1 are equal and the maximum in the quadrilateral (see above Proposition): σ is the maximum value of γ such that this condition holds. Another equation for the case when the angles at x and v_3 are the maximum is needed. The only change in the above construction comes by changing equation (1). We now have

$$r^2 = l_2^2 + l_3^2 - 2l_2l_3 \cos \alpha.$$

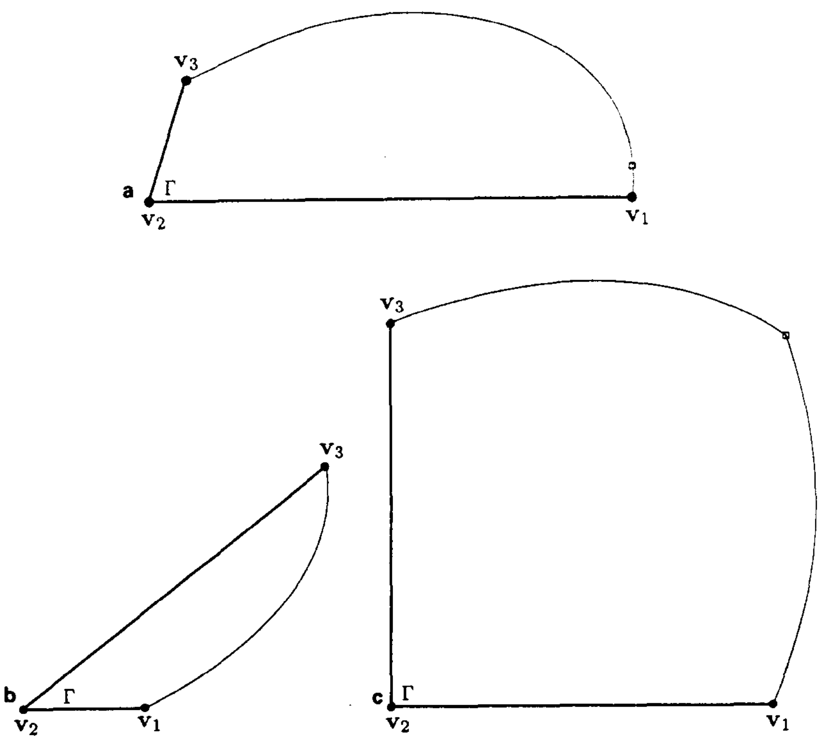

Fig. 4. Some examples of neutral sets using the min-max criterion. The box indicates a change from equation (5) to (6). (a) $\Gamma = 73°$, (b) $\Gamma = 38.16°$ (used equation (5) only), (c) $\Gamma = 90°$.

Equation (5) becomes

$$\frac{\left(-2l_2^2 + 2l_2 r \cos \delta\right)}{l_2} = \frac{\left(-l_1^2 - l_2^2 + l_5^2 - 2r^2 + 2l_2 r \cos \delta + 2l_1 r \cos \gamma\right)}{\left(l_1^2 + r^2 - 2l_1 r \cos \gamma\right)^{1/2}}. \tag{6}$$

It is not difficult to determine which of the two expressions we should use, (5) or (6), during the evaluation process. In Figs. 4(a)–(c) are some examples of the neutral set for various combinations of three vertices with $\Gamma \leqslant 90°$.

It is interesting to note that if Γ is restricted to ninety degrees, then equation (5) simplifies enough to convert from polar coordinates to Cartesian coordinates. The expression is now a fourth-order equation in terms of $\boldsymbol{x} = (x, y)^{\mathrm{T}}$:

$$\begin{aligned} &4x^2 l_1^2 l_2^2 - 8x l_2^2 l_1^3 + 8xy l_2 l_1^3 - 4x^2 y^2 l_1^2 - 8y l_2 l_1^4 + 4y^2 l_1^4 - 4y^2 l_2^2 l_1^2 + 8y^3 l_2 l_1^2 \\ &\quad - 4y^4 l_1^2 + 4 l_2^2 l_1^4 = 0 \end{aligned} \tag{7}$$

for $x \geqslant 2$ and $0 < y \leqslant 1$. To express equation (6) in cartesian coordinates for this special problem instances, we only need to do the following: switch x and y and switch l_1 and l_2 in equation (7) and now it may be applied for $y \geqslant 1$, $0 < x \leqslant 1$.

4. The solution to Problem 2

Unfortunately, the solution to this problem is not quite as simple as that for Problem 1. However, it is this solution that reveals more about the behavior of the min-max criterion.

Case 1 of this problem is characterized by the given angle $\Gamma = \alpha$. (Recall that, α is the magnitude of the two angles which are maximum and adjacent.) It is possible to construct the solution algebraically just as the solution to Problem 1, however more insight is gained by a geometric construction. In Fig. 2 we see that the other angle with magnitude α is located at $\boldsymbol{v}_1$. Using the angle at $\boldsymbol{v}_1$ and $\boldsymbol{v}_1$ we may define a line l; we know that $\boldsymbol{x}$ must lie on this line. Let us define β_1 and β_2 as the angles formed at $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ in the convex quadrilateral. The precise segment of l on which $\boldsymbol{x}$ may be located is such that β_1 and β_2 are less than or equal to α. We may also construct this case such that l has an endpoint at $\boldsymbol{v}_3$ and the magnitude of the angle at $\boldsymbol{v}_3$ is α. Notice, as in Fig. 5, that by considering the two constructions of the line l, the two parts of the neutral set may extend very far and do not necessarily intersect to form a closed

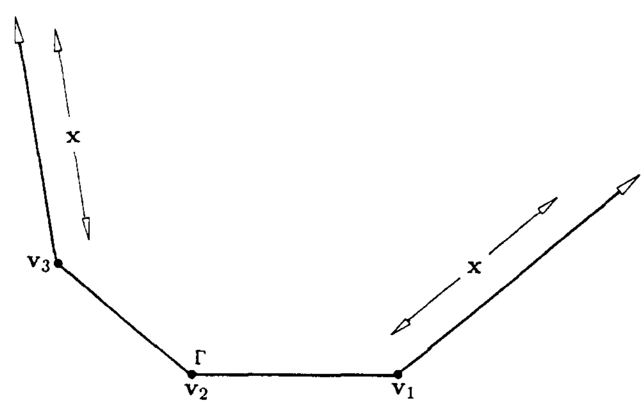

Fig. 5. Demonstrating that the neutral set does not necessarily form a closed region. The two parts of the neutral set live along both lines.

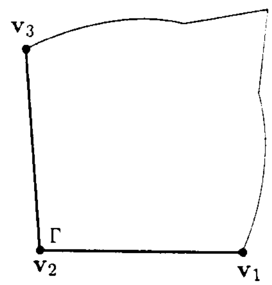

Fig. 6. $\Gamma = 95°$. Using the solutions to Problem 2, the neutral set is defined. Case 1 creates the straight lines that form an apex and case 2 creates the 'curved' lines.

region. The implications of this are discussed in Section 5. (This unbounded region and the closed regions as in Figs. 4(a)–(c) will be refered to as 'interior to the neutral set'.)

Case 2 is characterized by $\Gamma < \alpha$. Thus the angles with magnitude α may be as in Fig. 3 or the angle at v_3 may have magnitude α instead of the angle at v_1. Again, simple calculations reveal that $90° < \Gamma \leqslant 120°$. Just as with case 1, a better understanding of the neutral set created by this case is realized by a more geometric approach. Let us consider the situation as in Fig. 3.

First we must label another angle. Let ϵ be the angle formed at v_3 when considering the triangle formed by (v_1, v_2, v_3). The two angles with magnitude α must satisfy

$$\Gamma < \alpha < \frac{360° - \Gamma - \epsilon}{2}.$$

By being given v_1, v_2, and v_3 such that $\Gamma < \alpha$, we know ϵ and thus the range of values that α may take are determined. An example of this case is illustrated in Fig. 6 by the 'curved' lines. Notice that the shape of the neutral set from this case is similar to the sets created in the solution to Problem 1. Also illustrated in Fig. 6 is the neutral set created by case 1 (the straight lines that meet at an apex). This example demonstrated that if $90° < \Gamma < 120°$, then it is necessary to satisfy four conditions in order to find the complete neutral set (two subcases from case 1 and case 2).

5. Analyzing the solutions

It has been observed that in some cases the min-max and max-min criteria applied to a point set yield identical triangulations [Nielson & Franke '83, Piper '86]. This can most likely be attributed to the solution to Problem 1: $\Gamma \leqslant 90°$. By comparing, as in Fig. 7, the min-max neutral set for $\Gamma \leqslant 90°$ to the max-min neutral set, we see that they are different, however not drastically so. The region between the bold and thin lines is the loci of a fourth vertex that would produce different triangulations for the two criteria. In many instances, there is only a small region that would yield different triangulations. Fig. 7 also demonstrates that in general neither neutral set curve lies entirely interior to the other neutral set. As a result, it is difficult to completely determine the triangulation produced by one criterion given the triangulation constructed by application of the other criterion.

Turning to the solution of Problem 2 with $\Gamma \geqslant 90°$, there is much more to analyze. First we must recall an important finding regarding the max-min criterion. Lawson [Lawson '86] has shown for a globally optimal max-min triangulation, that by not having points interior to the neutral set it follows that a triangulation with the global property also has the local property. It has been shown through example in [Piper '86] that with the min-max criterion, a triangulation

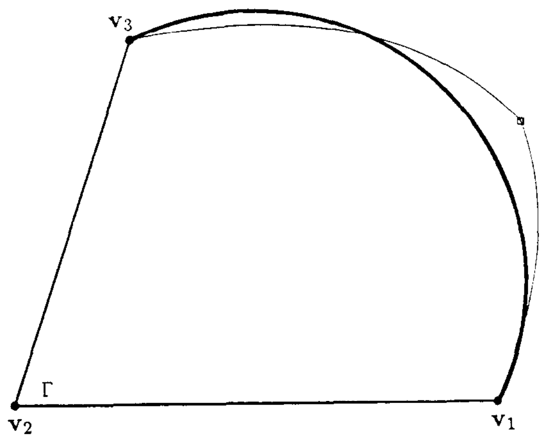

Fig. 7. The min-max (fine line) and the max-min (bold line) neutral sets. Demonstrating that neither neutral set is interior to the other.

with the global property does not imply a triangulation with the local property. This means that we should expect points to be interior to the min-max criterion's neutral set. This paper has demonstrated this to be the case using our geometric construction of the neutral sets for the min-max criterion. For instance, it was shown in Section 4 with Figs. 2 and 5 that the neutral set may create large and not always bounded regions (of course the regions would be bounded by the convex hull of the data set). Thus it must be possible in the globally optimal solution that there are data points interior to the neutral set. As a result, a local swapping algorithm as used for the max-min criterion will not in general produce a globally optimal triangulation. This observation implies a global algorithm is necessary in general. In addition, to compute the neutral set associated with a set of three vertices is not a trivial calculation as is the circle for the max-min.

6. Conclusions

We have developed conditions for constructing the neutral case for the min-max triangulation criterion. In addition, we have shown that the relationship between the min-max neutral set and the max-min neutral set is somewhat complex. The nature of the differences between these two criteria is revealed through the neutral set analysis. In particular, by examining the geometry of the neutral set, we can understand why a local swapping algorithm will not in general work for the min-max criterion.

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