

# Barycentric Coordinates

## Introduction to Computer Graphics

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## 1 Introduction

Triangles are an important primitive in computer graphics because they represent a planar face. Because of its simple form, calculations on a triangle sent down the rendering pipeline can be done very fast. Barycentric coordinates are a fundamental tool for dealing with triangles. For example, barycentric coordinates make the following two problems easy to solve.

*Triangle inclusion test:* Given a triangle and a point  $\mathbf{p}$  in the plane formed by the triangle, is the point in the triangle? This scenario arises in the rasterization step: Is a given pixel within the boundaries of a rasterized triangle?

*Linear interpolation over a triangle:* Given a triangle and function values at the vertices, determine a function value at any point in the triangle. For example, the function values might be colors, and then barycentric coordinates help us fill-in meaningful colors in the interior of the triangle. This is called *shading*.

## 2 Barycentric coordinates with respect to a line

*Given:* a line defined by points  $\mathbf{p}_1$  and  $\mathbf{p}_2$ .

*Find:* a point  $\mathbf{p}$  on the given line.

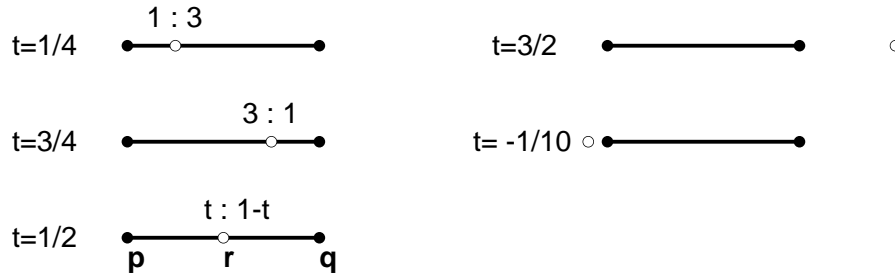


Figure 1: Several examples of points  $\mathbf{p}$  with parameter  $t$  on a line defined by points  $\mathbf{p}$  and  $\mathbf{q}$ .

*Solution:*

$$\mathbf{p} = (1 - t)\mathbf{p}_1 + t\mathbf{p}_2. \quad (1)$$

The *parameter*  $t$  may range  $[-\infty, \infty]$ , however for  $t \in [0, 1]$  we trace out points between  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . Some examples are illustrated in Figure 1

*Definitions:*

*Barycentric combination:* A weighted sum of points where the coefficients of the points sum to one. Equation (1) is an example.

*Barycentric coordinates:* These are the coefficients in a barycentric combination. In Equation (1),  $t$  and  $(1 - t)$  are the barycentric coordinates.

Notice that we have set up a coordinate system on the line.

### 3 Barycentric coordinates with respect to a triangle

A triangle is given by three points,  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ , its vertices. The vertices may live in 2D or 3D. Three points define a plane, therefore, a triangle is a 2D element. For the following development, let's suppose the triangle is given in the  $xy$ -plane.

A barycentric combination of three points takes the form:

$$\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3, \quad \text{where } u + v + w = 1. \quad (2)$$

Therefore,  $\mathbf{p}$  lies in this plane, and we say  $[u, v, w]$  are the barycentric coordinates of  $\mathbf{p}$  with respect to  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ , respectively.

Equivalently, we may write

$$\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + (1 - u - v)\mathbf{p}_3.$$

How can we find the barycentric coordinates of a given point  $\mathbf{p}$ ? Equation (2) represents three equations, and thus we can form a linear system

$$\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}$$

for the unknown  $u, v, w$ . Solve this system using Cramer's rule, and recall that determinants correspond to *signed* areas:

$$\begin{aligned} A &= \begin{vmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{vmatrix} \\ A_1 &= \begin{vmatrix} \mathbf{p} & \mathbf{p}_2 & \mathbf{p}_3 \\ 1 & 1 & 1 \end{vmatrix} \\ A_2 &= \begin{vmatrix} \mathbf{p}_1 & \mathbf{p} & \mathbf{p}_3 \\ 1 & 1 & 1 \end{vmatrix} \\ A_3 &= \begin{vmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p} \\ 1 & 1 & 1 \end{vmatrix}. \end{aligned}$$

Therefore,

$$u = A_1/A \quad v = A_2/A \quad w = A_3/A. \quad (3)$$

Thus the barycentric coordinates are simply ratios of triangle areas. Area  $A$  is simply the triangle's area, and the "sub-areas" are illustrated in Figure 2. These ratios of areas form dimensionless coordinates in the plane defined by  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ . It is more efficient to calculate  $w = 1 - u - v$ .

We see that  $u, v, w$  do indeed sum to one. Next check some examples:

- The three vertices have barycentric coordinates

$$\mathbf{p}_1 \cong [1, 0, 0] \quad \mathbf{p}_2 \cong [0, 1, 0] \quad \mathbf{p}_3 \cong [0, 0, 1]$$

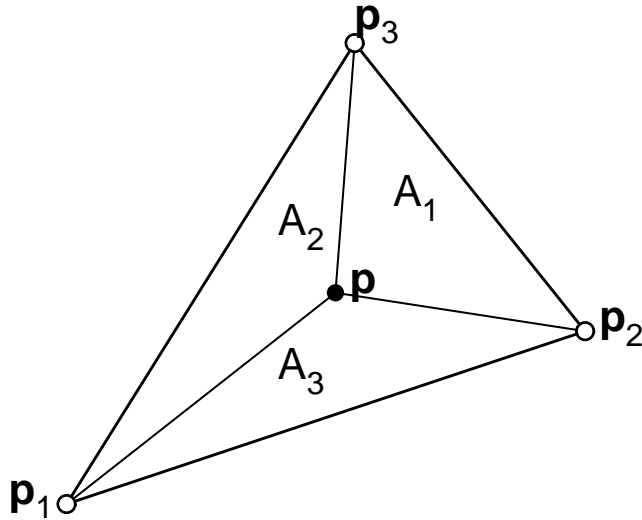


Figure 2: The barycentric coordinates of a point  $\mathbf{p}$  with respect to a triangle are defined in terms of ratios of triangle areas.

- Points inside the triangle have positive  $u, v, w$ .
- Points outside the triangle have at least one negative coordinate. Recall that a determinant can be negative – in fact, the points forming the column vectors of the matrix of which we take the determinant must be counterclockwise for the determinant to be positive. This gives us *signed area*.
- The barycenter of the triangle has coordinates  $[1/3, 1/3, 1/3]$ .
- A point which is the midpoint of  $\mathbf{p}_1$  and  $\mathbf{p}_2$  has coordinates  $[1/2, 1/2, 0]$ .

Notice that barycentric coordinates are simply another coordinate system for the plane!

## 4 A 3D Triangle

Now let's look at the situation when the  $\mathbf{p}_i$  live in 3D. Suppose we are looking at the triangle's "front" face, and thus by convention the labelling of the  $\mathbf{p}_i$  will be *counterclockwise*; This follows the right-hand rule.

Form the vectors  $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$  and  $\mathbf{w} = \mathbf{p}_3 - \mathbf{p}_1$ . The cross product of these two vectors results in a vector:

$$\mathbf{u} = \mathbf{v} \wedge \mathbf{w},$$

and  $\mathbf{u}$  is perpendicular to  $\mathbf{v}$  and  $\mathbf{w}$ . The area of the parallelogram formed by  $\mathbf{v}$  and  $\mathbf{w}$  is  $A = \|\mathbf{u}\|$ .

Now we can find the barycentric coordinates of a point  $\mathbf{p}$  in the plane formed by the  $\mathbf{p}_i$ . We simply use this same method to form the sub-areas  $A_1, A_2, A_3$  in (3). The triangle area is simply one-half the parallelogram's area, but since we plane to use ratios of areas, we need not divide each by two.

However, we need to take some care to reintroduce the concept of signed area. First, consider the vector  $\mathbf{u}$  that we formed to calculate  $A$ . For the sub-area calculations, form  $\mathbf{u}_i$  so to preserve the orientation of the points as indicated in the determinants in the previous section. Taking the dot product,

$$s = \mathbf{u} \cdot \mathbf{u}_i$$

results in a value that determines the sign of the area. If  $s > 0$  then the area  $A_i/A$  is positive; if  $s < 0$  then the area is negative.