Notation, Point & Vector Basics
Introduction to Computer Graphics
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1 Notation

Let's focus on 2D. After that, 3D is easy!

The $[\mathbf{e}_1, \mathbf{e}_2]$-coordinate system is defined by the origin $\mathbf{o}$, and the $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors:

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A *point* $\mathbf{p}$ describes a location and its coordinates are defined as

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

Therefore, a point's location is defined by its coordinates in this coordinate system: $\mathbf{p} = \mathbf{o} + p_1\mathbf{e}_1 + p_2\mathbf{e}_2$. Sometimes it is convenient to label the coordintes of $\mathbf{p}$ as p_x and p_y.

A *vector* $\mathbf{v}$ describes a displacement (direction and distance) and its components take the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

and thus $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$. Figure 1 illustrates this basic geometry.

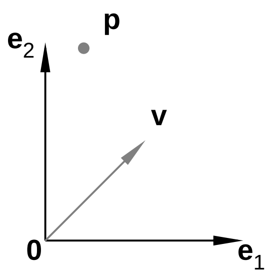

Figure 1: Point and vector notation.

2 Point and Vector Combinations

As illustrated in Figure 2, the sum of a point and a vector is a point:

$$\mathbf{q} = \mathbf{p} + \mathbf{v}. \tag{1}$$

Vector addition may be described in terms of the *parallelogram rule*, as illustrated in Figure 3. Therefore, any scalar values s and t are acceptable in forming $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$.

To formulate point addition, let's revisit (1) and the idea that a point plus a vector results in a point. As in Figure 4, we may form a point on the line through $\mathbf{p}$ and $\mathbf{q}$ as

$$\begin{aligned} \mathbf{r} &= \mathbf{p} + t\mathbf{v} && (2)\\ &= \mathbf{p} + t(\mathbf{q} - \mathbf{p}) && (3)\\ &= (1-t)\mathbf{p} + t\mathbf{q}, && (4) \end{aligned}$$

therefore, the coefficients for point addition must sum to one. Such a requirement on the coefficients is called a *barycentric combination.* Another name for (4) is *linear interpolation.* Notice that when the *parameter* $t = 0$, $\mathbf{r} = \mathbf{p}$, when $t = 1$, $\mathbf{r} = \mathbf{q}$, and when $t = 1/2$, $\mathbf{r}$ is the midpoint between $\mathbf{p}$ and $\mathbf{q}$.

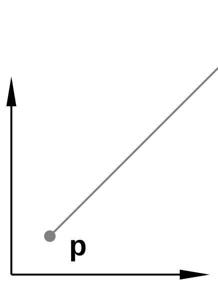

Figure 2: The sum of a point and vector yields a point.

Thus we see that the relative positioning of $\mathbf{p}, \mathbf{q}, \mathbf{r}$ may be characterized by the parameter

$$t = \frac{||\mathbf{r} - \mathbf{p}||}{||\mathbf{q} - \mathbf{p}||},$$

using a signed distance, as illustrated in Figure 5. The relationship between the three points is also reflected in the quotient

$$\text{ratio}(\mathbf{p}, \mathbf{r}, \mathbf{q}) = \frac{t}{1-t} = \frac{||\mathbf{r} - \mathbf{p}||}{||\mathbf{q} - \mathbf{r}||},$$

and this ratio is also illustrated in Figure 5.

Point addition comes with some rules for two reasons.

1. We would like coordinate-system independent operations. For example, as illustrated in Figure 6, averaging two points should always yield the same point.

2. We want a method to construct a new point within the confines of the geometry defined by the given points. For example, as illustrated in Figure 4, we would like to construct a point on the line defined by $\mathbf{p}$ and $\mathbf{q}$.

For more information, examples, and exercises, see **Practical Linear Algebra – A Geometry Toolbox by G. Farin & D. Hansford, A K Peters, 2005**.

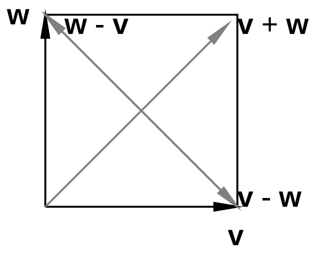

Figure 3: The parallelogram rule encapsulates vector addition.

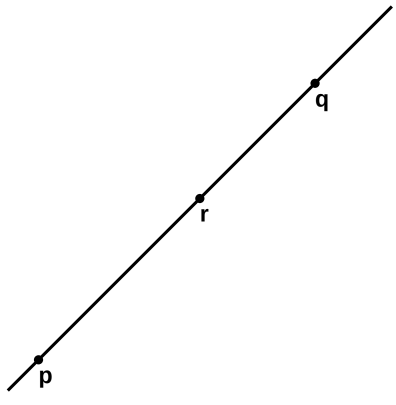

Figure 4: The operation $\mathbf{r} = (1 - t)\mathbf{p} + t\mathbf{q}$ results in a point on the line defined by $\mathbf{p}$ and $\mathbf{q}$.

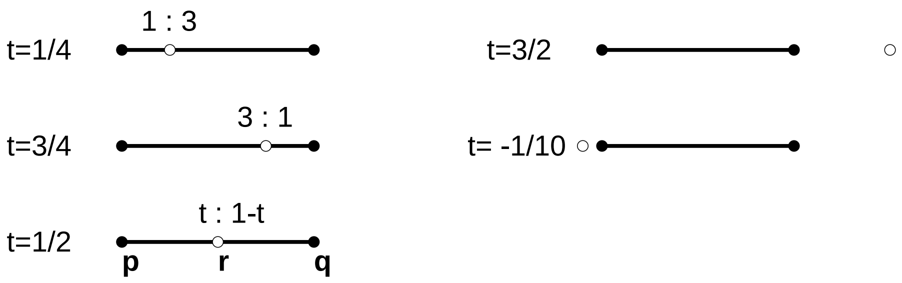

Figure 5: Examples of the ratio of three points and the corresponding parameter t.

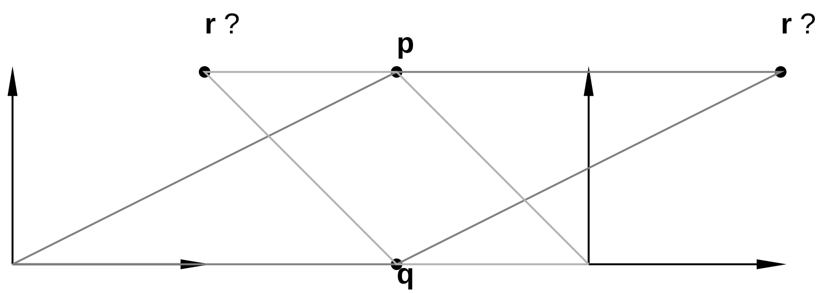

Figure 6: The result of the operation $\mathbf{r} = \mathbf{p} + \mathbf{q}$ is dependent on the coordinate system.