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About Numbers

1.1 A BIT OF HISTORY

The origins of man's concept of number are hidden from us in the obscure prehistoric past and will never be fully known. Whatever speculations we may make about the way that early man may have gradually formulated the idea of number as a characteristic separate from particular numbers of particular objects, they remain speculations. Our guesses about this prehistory are made at least reasonable by the fact that there still exist certain isolated tribal groups (usually described in newspaper accounts as having a "stone age" culture) in which only a few, if any, number names exist and in which separate and distinct names are used for such different entities as "two fish," "two men," or "two spears." In these languages no word exists for the abstract concept "two," and we may well suspect that prehistoric man once struggled with similarly limited languages. If the language does not include a means for naming numbers, it seems self-evident that it cannot include descriptive terms for arithmetic processes, that is, for the processes of generating the numbers implied by operations on given numbers.*

The first records we have of the use of numbers date from the ancient Sumerian civilization of Mesopotamia. The Sumerians and their successors, the Babylonians, left tens of thousands of baked clay tablets inscribed with records of commercial transactions in the period 4000–1200 B.C. These tablets do not reveal anything about the de-

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*The extent to which language influences thought processes has been widely examained by linguists and psychologists since the American linguist Benjamin Lee Whorf made his classic studies of the Hopi language in the 1920s and concluded that one's language is a principal factor in determining how one thinks. That is, how we think about things is determined by what our language enables us to say about them. Although the Whorfian thesis has not been universally accepted, it does appear to apply in the case of numerical reasoning.

Very interesting!

velopment of the number concept, since they display a rich, well-developed, and widely used number system, one already in full flower rather than one struggling for definition. Although we might have expected this first recorded number system to be only a rudimentary one, it is surprisingly well developed and was the instrument of an advanced system of commerce. Like our present-day decimal system, the Sumerian system was a positional and based system; that is, digit position indicated what power of the base was to be multiplied by the digit value in evaluating the number. Unlike our system, the base was 60, and furthermore there was no symbol for zero. When no units in a particular position were to be specified, the ancient scribes and readers of records had to rely on contextual aids to indicate the positional significance of the other digits. It is now fairly well agreed that our symbol for zero came from India, reaching the Arabic world around the ninth or tenth century A.D., and finding its way thence to the Western world. (It is interesting to note the great debt owed by the Western world to the Moslem world, for at the very time that Europe was struggling through the centuries of the Dark Ages, Moslem culture was in full bloom, and the Moslem world was a repository of civilization.) Use of the symbol "zero" made possible much simpler representations of numbers than did other systems, such as Roman numerals.

Another feature of modern number systems that was not present in ancient times (or even in fairly recent times) was the concept of negative numbers. Although Bell, in his fascinating book on the history of mathematics [1], cites evidence that the Babylonians accepted negative numbers as well as positive numbers in solving pairs of simultaneous equations, the idea apparently did not take hold. Bell remarks: "The one glimmer of mathematical intelligence in the early history of negatives is the suggestion of Fibonacci that a negative sum of money may be interpreted as a loss." Since Fibonacci lived in the period around A.D. 1200, we can see how recently the concept of negative numbers has come into use.

Roman numerals - limitations

1.2 NAMES FOR NUMBERS

We have already noted that primitive languages exist that have few words or symbols for representing numbers. On the other hand, the diversity of number names and symbols among various languages and cultures is very large. For example, the number that in English is called "nine" is recognized by other cultures and other ages by such names or symbols as

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and many others. Yet each culture tends to limit its set of number names to those that are frequently needed in the daily affairs of its society. A society of sheepherders has no occasion to need the word "million," and "billion" and its successors have come into use in the English language only within the last century. "Trillion" finds use nowadays only in measuring that ill-defined quantity called the gross national product, al-

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though it seems likely that within a few more years Americans will learn to use it when speaking of the annual federal budget. "Quadrillion" and similarly formed names for bigger numbers begin to grow clumsy, and scientists and engineers prefer to use scientific notation, which uses exponents of 10 to indicate magnitudes.

We should note at the start that merely assigning names to numbers is not sufficient to define a number system. Since the number of numbers is infinite, we cannot arbitrarily assign names to all numbers but must use a recursive definition in which the names of most numbers will be expressed as combinations of names of other numbers. For example, the number name "six hundred" uses the names of two other numbers. Of course, the names themselves are usually used only in spoken communication, and for writing we use symbols of various kinds. In the computer we use yet other representations, such as voltage levels, states of magnetization, or combinations of holes or no holes.

It is interesting to observe that in English the unique number names grow sparser as the numbers grow bigger, and that to name larger numbers we are increasingly dependent on the recursive use of names of smaller numbers. Thus the first 21 numbers, 0 through 20, have their own individual names (although some of them originated as compound names). To name 21 through 29, we use combinations of other names. Then comes another unique name, "thirty" (again originally a compound word). Every tenth name beyond has a unique name up through "hundred." Then follow only compound names up to "thousand," again only compound names up to "million," and so on. As is so often the case in both natural and artificial languages, we find short symbols being used for frequently occurring "messages."

The distinguished Argentinian writer Jorge Borges, many of whose writings display his fascination with numbers, describes in the story "Funes the Memorious" [2] a strange "system" of number names invented by the mystic Funes:

> He told me that in 1886 he had invented an original system of numbering and that in a very few days he had gone beyond the twenty-four-thousand mark. He had not written it down, since anything he thought of once would never be lost to him. His first stimulus was, I think, his discomfort at the fact that the famous thirty-three gauchos of Uruguayan history should require two signs and two words, in place of a single word and a single sign. He then applied this absurd principle to the other numbers. In place of seven thousand thirteen, he would say (for example) *Maximo Perez*; in place of seven thousand fourteen, *The Railroad*; other names were *Luis Melian Lafinur, Olimar, sulphur, the reins, the whale, the gas, the caldron, Napoleon, Agustin de Vedia*. In place of five hundred, he would say *nine*. Each word had a particular sign, a kind of mark; the last in the series were very complicated . . . I tried to explain to him that this rhapsody of incoherent terms was precisely the opposite of a system of numbers. I told him that saying 365 meant saying three hundreds, six tens, five ones, an analysis which is not found in the "numbers" *The Negro Timoteo* or *meat blanket*. Funes did not understand me or refused to understand me.*

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*Jorge Luis Borges, LABYRINTHS. Copyright © 1962 by New Directions Publishing Corporation, Reprinted by permission of New Directions.

We may note also that since earliest times numbers have possessed a mystical significance for some people. For example, the Roman writer Varro (116–27 B.C.) felt that "the virtues and powers of the number 7 are many and various," and therefore wrote 70 times 7 books and a set of 700 biographies [3]. Countless other examples of this nonsense exist, which is harmless in such innocent forms as omitting 13 when numbering the floors of hotels or office buildings but can become dangerous anti-intellectualism when it becomes popularized in such forms as the present-day fad of astrology.

1.3 HOW BIG ARE NUMBERS?

Let us examine for a moment the matter of how big numbers have to be to encompass the things we want to state numerically. Archimedes set the style for such investigations in his work "The Sand Reckoner" [4] by setting out to compute how many of the smallest elements known would be needed to fill up the biggest space known, choosing for the smallest element a grain of sand and for the largest space the universe as then known. The number involved was so far beyond the size of any needed in daily life of that time (the third century B.C.) that Archimedes was obliged to invent a new way of expressing very large numbers. He started with the Greek number "myriad" = 10,000, and noting that a myriad myriads can be expressed, he termed all lesser numbers "numbers of the first order." Then taking 10^8 as the unit size for the next range, numbers from that size to $(10^8)^2$ were termed "numbers of the second order." He similarly defined the third, fourth, and fifth orders, up through the 10^8 order, with the number $P = 10^{8 \times 10^8}$ being the largest number of this first period of numbers. The second period started from P, its first order extending to 10^8P, its second order to $(10^8)^2P$, and its 10^8 order ending with $10^{8 \times 10^8}P$, that is, P^2. Continuing the process to the 10^8 order of the 10^8 period reaches at last the truly big number P^{10^8}, that is, $(10^{8 \times 10^8})^{10^8} = 10^{8 \times 10^{16}}$ = the digit 1 followed by 8×10^{16} zeros. Using this notation, he then went on to calculate the answer to his original problem, finding the number of grains of sand that could be encompassed by the universe to be less than 1000 units of the seventh order of numbers, that is, less than 10^{52}.

A more recent exploration of big numbers was described by the American mathematician Warren Weaver in an article in the *Atlantic Monthly* [5]. By this time, both the smallest particles and the universe were better defined. Weaver calculated the ratio of the diameter of the universe to the diameter of the proton to be about 10^{42}, a very big number by comparison with those that most of us meet in our daily affairs, but mathematically almost pitifully small. Weaver similarly found the age of the universe, when expressed in terms of the smallest meaningful time unit (which he took as 10^{-30} second, the period of the highest-energy cosmic rays known), to be 10^{47}, another big but not inconceivable number. His candidate for "biggest meaningful number" was one that he attributes to "an English mathematician named Skewes [who] in connec-

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tion with a theorem about prime numbers found specific use for what we will call the Skewes number, namely:

$$10^{10^{10^{34}}}$$

This symbol stands for the utterly fantastic number which, if written out in full, would have

$$10^{10^{34}} = 10^{10,000,000,000,000,000,000,000,000,000,000,000}$$

zeros."

The latest such exploration was one conducted by D. E. Knuth, that Renaissance Man of computer mathematics, and reported by him in *Science* magazine for December 17, 1976 [6]. Knuth also briefly investigates some "astronomically" large numbers, but like Weaver, finds that the truly "big" numbers arise in problems in number theory and in particular those that involve combinatorial analysis. Following Archimedes' lead, Knuth also finds it expedient to introduce a notation for expressing large numbers. As are all good notations, his is both concise and simple:

1. We let $x \uparrow n$ denote n factors x multiplied together (which we ordinarily write as x^n).
2. By simple extension of this notation, $x \uparrow\uparrow n$ denotes $x \uparrow (x \uparrow (x \uparrow (\cdots \uparrow x) \cdots))$, where we take powers n times.
3. The general rule is

$$x \overbrace{\uparrow\uparrow \cdots \uparrow}^{k \text{ arrows}} n = \underbrace{x \overbrace{\uparrow\uparrow \cdots \uparrow}^{k-1} (x \overbrace{\uparrow\uparrow \cdots \uparrow}^{k-1} (x \overbrace{\uparrow\uparrow \cdots \uparrow}^{k-1} (\cdots \overbrace{\uparrow\uparrow \cdots \uparrow}^{k-1} x)) \cdots)}_{n \text{ times}}$$

Like most number system definitions, this one also is recursive, in that each group of arrows is defined in terms of groups having one fewer arrows, down to the terminating level $x \uparrow n = x^n$. It is also much more concise than the Archimedian notation, and it is interesting to note that Archimedes' biggest number is much smaller than the number $10 \uparrow\uparrow 2$. In a similar manner, even the "utterly fantastic" Skewes number is much smaller than $10 \uparrow\uparrow 4$, which is a very small number in Knuth's notation.*

Of course, it is not the purpose of the preceding discussion to demonstrate that computer hardware should be capable of representing enormously big numbers. Whatever the number range the designer elects to build into the hardware, the programmer has the liberty by suitable choice of scaling factors to use that number range for the representation of other number ranges. One part of our purpose is simply to demon-

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*Indeed, a vastly bigger number than the Skewes number has been found by Ronald Graham of Bell Laboratories, as an upper bound on a certain combinatorial problem. It is described in *Science*, vol. 218, Nov. 19, 1982, p. 780.

strate that there are enormously many numbers "out there" to be manipulated by our computers, and the system designer is confronted by a very real challenge in trying to devise computer number representations and arithmetic systems that will be suitable for the ones most frequently encountered and at least usable on the occasional exceptional number.

1.4 THE USES AND MISUSES OF NUMBERS

Numbers play an enormously important role in today's advanced technological societies, which could not exist without them. We use numbers to express units of trade and commerce, monetary values, physical dimensions—indeed, to express the measure of any concept that can be quantified. One hundred years ago, the eminent scientist-engineer Lord Kelvin said: "In physical science a first essential step in the direction of learning any subject, is to find principles of numerical reckoning, and methods for practicably measuring, some quality connected with it. I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind: it may be the beginning of knowledge, but you have scarcely in your thoughts, advanced to the stage of *science*, whatever the matter may be" [7].

The need to quantify and to assign numerical values has been carried so far that we are awash every day in a sea of numbers—weather reports, baseball statistics, election returns, stock market reports, grocery prices, calorie counts, census data, and a thousand other aspects of today's world. Indeed, many of the concepts to which we attach numbers are either intrinsically not quantifiable, or are precise concepts that are not widely understood, or are statistical concepts that are neither well defined nor widely understood. The reader has only to examine a daily newspaper to encounter references to batting averages, the consumer price index, the Dow Jones average, the gross national product, and public opinion polls predicting election outcomes, among other numerical measures, and it is an unusual reader who knows how all these numbers are calculated, what they signify, and how trustworthy they are. Lacking understanding of the source and meaning of these numerical quantities, most people are content to accept them reverently and to cite them glibly when the occasion arises. The urge to quantify and to number so dominates modern conversation and writing that no topic seems adequately treated without some numerical reference, and the most significant aspect of a great work of art may be the price it fetches at auction.

Although our entire culture blithely bandies numbers about, there is a widespread ignorance of numerical magnitudes and an equally broad incompetence at simple arithmetic. It has often been remarked that skill at arithmetic is unnecessary now that everyone has two or three pocket calculators, but the mere existence of a device to calculate does not imply that a user will know how to employ it, or what to do if the batteries wear out. The inability to understand letters, that is, to read, is known as

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"illiteracy," and we may similarly speak of "innumeracy"[*] to describe the inability to use numbers. This condition is far more general than might be suspected by computer people who talk only to computer people, and in today's technological world, there should be as much concern over "numeracy" as over literacy. The "two cultures" may prove to be those who can and those who cannot use numbers.

1.5 DEFINITION OF "NUMBER": PEANO'S AXIOMS

In the foregoing pages we have examined some of the historical and cultural aspects of numbers and the concept of "number" without attempting to define just what "number" might be. For most purposes of ordinary daily life, "number" is just another commonly accepted word of our customary language and is well understood by all of us without any more need for precise definition than any other word, such as "other" or "word." However, as we begin to investigate algorithms for manipulating numbers and hardware structures for implementing these algorithms, it is essential to have a clear understanding of what we mean by "number." This clear understanding is not at all facilitated by the fact that in the English language (at least), there is no clear distinction between "number" as an abstract concept and "number" as a set of symbols or a name for a particular instance of this concept. Thus "23" is said to be a *number*, and "twenty-three" is the name of that number. However, in other languages and other cultures, other symbols and other names are used. Behind the various symbols and names is the concept of "number," which is shared by all of them.

To define this abstract notion of "number," we make use of *Peano's axioms* defining the positive integers, that is, the set of "natural numbers":

> The positive integers are the set of elements that satisfy the following postulates:
>
> 1. There is a positive integer 1 in the set.
> 2. Every positive integer a has a distinct successor positive integer a^+.
> 3. The element 1 is not the successor of any positive integer.
> 4. If two elements have the same successor, then they are the same element. That is, if $a^+ = b^+$, then $a = b$.
> 5. Every set of positive integers that contains 1 and the successor of every element of the set contains all the positive integers.

We may note several points about this set of axioms:

1. As do all axiom sets, it contains some undefined words and terms, for whose meaning we rely on the commonly accepted interpretations.

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[*] I wish I had invented this felicitous term, but credit goes to an English writer (whose name I don't recall), writing in the periodical *New Society* sometime in the 1960s.