An Introduction to Logic

§ 1.1 ~ § 1.4

6/21/04 ~ 6/23/04

A Taste of Logic

- Logic puzzles

  (1) Knights and Knaves

  Knights: always tell the truth   Knaves: always lie

  You encounter two people A and B.

  A says: "B is a knight"

  B says: "The two of us are opposite types"

  Q: What are A and B?

  (2) Murder suspects

  3 suspects for murdering Cooper: Smith, Jones, Williams

  Smith: "Cooper is a friend of Jones, and Williams dislikes him"

  Jones: "I don't know Cooper and I am out of town that day"

  Williams:"I saw both Smith and Jones with Cooper that day, and either Smith or Jones must have killed him"

  Q: Who is the murder?

What is Logic?

- *Mathematical Logic* is a tool for providing precise meaning to mathematical statements. It includes:
  - A formal language for expressing them.
  - A concise notation for writing them.
  - A methodology for objectively reasoning about their truth or falsity.
- Applications of Logic
  - It is the foundation for expressing formal proofs in all branches of mathematics.
  - In computer science
    - Circuit design
    - Program constructing
    - Program verification
    - …

§ 1.1 Logic

- DEF1: A ***proposition*** is a declarative sentence that is either true (T) or false (F), but not both.
- Ex1: propositions

  1) Beijing is the capital of China

  2) 2 + 3 = 5

  3) 2 + 3 = 4
- Ex2: not propositions

  1) Is your major computer science? (question)

  2) Finish the homework by yourself. (command)

  3) What a smart student! (exclamation)

  4) x + 1 = 3 (could be both T and F)
- Exercise1: proposition or not?

  1) You are a student.

  2) Lingli is a cute girl.

  3) There is life on Mar.

  4) x + 1 = 5 if x = 1

  5) 2 + 3

Logical operators

- DEF2: A ***proposition variable*** is a variable such as p, q, r over the boolean domain {T,F}
- DEF3: A ***logical operator*** is a rule defined by a truth table
- Some popular logical operators

Operator	**Nickname**	**notation**
Negation	NOT	$\neg$
Conjunction	AND	$\wedge$
Disjunction	OR	$\vee$
Exclusive-OR	XOR	$\oplus$
Implication	IMPLIES	$\rightarrow$
Biconditional	IFF	$\leftrightarrow$

The Negation Operator

- Expression:
  - "It is not the case that p"
  - "p is false"
- Notation: ¬ p
- Truth table:

p	$\neg p$
T	F
F	T

- Ex3:
  - p = "Today is Friday"
  - ¬ p = "Today is not Friday", or "It is not the case that today is Friday

The Conjunction Operator

- Expression:
  - "p and q"
  - "Both p and q are true"
- Notation: $p \wedge q$
- Truth table:

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

- <u>Ex4</u>:
  - p = "Today is Friday"
  - q = "It is raining today"
  - $p \wedge q$ = "Today is Friday and it is raining today"

The Disjunction Operator

- Expression
  - "p or q"
  - "Either p or q, or possible both are true"
- Notation: $p \vee q$
- Truth table:

p	q	$p \vee q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

- Ex5:
  - p = "Today is Friday"
  - q = "It is raining today"
  - $p \vee q$ = "Today is Friday or it is raining today"
- Note: Disjunction is also called "inclusive or"

The Exclusive OR Operator

- Expression:
  - "Either p or q is true, but not both are true"
- Notation: $p \oplus q$
- Truth table:

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

- Ex6
  - p = "I will marry Helen"
  - q = "You will marry Helen"
  - $p \oplus q$ = "Either you or I will marry Helen"

English "or" is ambiguous

- Ex7:
  - "Tomorrow will be sunny or partially cloudy"
  - Inclusive? Exclusive?
- Ex8:
  - "Joe's major is computer science or chemical engineering"
  - Inclusive? Exclusive?
- Note:
  - We can infer the meaning of English "or" based on context, but not always
  - Logic is more precise than natural languages
- Convention:
  - In this class, "or" in default is inclusive

The Implication operator

- Expression:

  "If p then q", "if p, q", "p implies q",

  "q if p", "q whenever p", "q follows from p"

  "q is necessary for p", "p is sufficient for q",

  "p only if q",
- Notation: $p \rightarrow q$
- Truth table:

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	**F**
T	T	T

  - Note:
    - $p \rightarrow q$ is F only when p is true and q is false.
    - q can be true even if p is false
    - $p \rightarrow q$ does not require that p or q is true

The Implication operator

- <u>Ex9</u>:
  - p = "Today is Friday"
  - q = "2 + 3 =6"
  - $p \rightarrow q$ = "If today is Friday, then 2+3=6", T or F?
- <u>Ex10:</u>
  - "1=0" $\rightarrow$ "pigs can fly", T or F?
  - "1=1" $\rightarrow$ "pigs can fly", T or F?
- Related implications of $p \rightarrow q$:
  - Negation: $\neg (p \rightarrow q)$
  - Converse: $q \rightarrow p$
  - Contrapositive: $\neg q \rightarrow \neg p$
  - Inverse: $\neg p \rightarrow \neg q$
- <u>Note</u>: only the contrapositive of $p \rightarrow q$ has the same truth value as $p \rightarrow q$

The Biconditional Operator

- Expression:
  - "p if and only if q"
  - "p is necessary and sufficient for q"
- Notation: $p \leftrightarrow q$
- Truth table:
  - Note:
    - this truth table is exactly the opposite of $\oplus$'s
    - $p \leftrightarrow q \equiv \neg (p \oplus q)$

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

- Ex11
  - p = "You can take the flight"
  - q = "You buy a ticket"
  - $p \leftrightarrow q$ = "You can take a flight if and only of you buy a ticket

Precedence of Logical Operators

- Ex12: $\neg r \land s \rightarrow q$
  - $(\neg (r \land s)) \rightarrow q$?
  - $(\neg r) \land (s \rightarrow q)$?
  - $\neg (r \land (s \rightarrow q))$?
  - $((\neg r) \land s) \rightarrow q$?
- Precedence
  - Order from high to low:

    $\neg, \land, \lor, \oplus, \rightarrow, \leftrightarrow$
  - $\land$ and $\lor$ are left associative:

    $p \land q \land r \equiv (p \land q) \land r$

    $p \lor q \lor r \equiv (p \lor q) \lor r$

Logical Operators

Operator	Nickname	Notation	Precedence
Negation	NOT	$\neg p$	1
Conjunction	AND	$p \wedge q$	2
Disjunction	OR	$p \vee q$	3
Exclusive-OR	XOR	$p \oplus q$	4
Implication	IMPLIES	$p \rightarrow q$	5
Biconditional	IFF	$p \leftrightarrow q$	6

Compound Propositions

- DEF4: An ***atomic propositional form*** is either a boolean constant or a propositional variable.
- DEF5: An ***compound propositional form*** is constructed by combining atomic propositional forms using logical operators
- Ex13: Some compound propositional forms on two variables:

  $\neg p, p \vee q, p \wedge q, p \oplus q,$

  $p \rightarrow q, p \leftrightarrow q, (p \vee \neg q) \rightarrow q, \ldots$
- Ex14: How can this English sentence be translated into a logical expression?

  "*You can access the Internet from campus only if you are a computer science major or you are not a freshman.*"

Analyze compound propositions

- Ex15:
  - Q: Analyze "p ∧ ¬ (r → ¬ q)" using truth table
  - A:

p	q	r	$\neg q$	$r \rightarrow \neg q$	$\neg(r \rightarrow \neg q)$	$p \wedge \neg(r \rightarrow \neg q)$
F	F	F	T	T	F	F
F	F	T	T	T	F	F
F	T	F	F	T	F	F
F	T	T	F	F	T	F
T	F	F	T	T	F	F
T	F	T	T	T	F	F
T	T	F	F	T	F	F
T	T	T	F	F	T	T

Logic and Bit Operations

- Bit: 0 or 1
- Represent truth values using bit: 1--T, 0--F
- Bit operators correspond to the logical operators by replacing T by 1 and F by 0
- Bit operations can be extended to operate on bit strings: (bit-wise)

01 1011 0110	
11 0001 1101	
11 1011 1111	Bit-wise OR
01 0001 0100	Bit-wise AND
10 1010 1011	Bit-wise XOR

Logical Puzzles

- Ex16: Knights and Knaves

  Knights: always tell the truth

  Knaves: always lie

  You encounter two people A and B.

  A says: "B is a knight"

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  Q: What are A and B?

Logical Puzzles

- Exercise2: Murder suspects

  3 suspects for murdering Mr. Cooper: Smith, Jones, Williams. All of them denied the murdering.

  Smith said: "Cooper is a friend of Jones, and Williams dislikes Cooper."

  Jones said: "I don't know Cooper and I am out of town that day."

  Williams said:"I saw both Smith and Jones with Cooper that day, and either Smith or Jones must have killed him"

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