Set Issues

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January 21, 2007

The naive definition of a set (using a list of elements, or a description of its members based on some property) is not without problems; in fact, it is not a consistent definition.

1 The Village Mechanic Dilemma

Let us start with a made-up story. A car mechanic is hired by a small western village. His job is to fix cars for all those folks who don't fix their own car. He is reimbursed by the village once a month, upon showing the accountant receipts of his efforts. This works well for years, until the mechanic's own car needs serious attention. He fixes it, and submits a claim. The following dialog ensues.

Accountant:	You have a claim for your own car?
Mechanic:	Yes, it broke down so I had to fix it.
Accountant:	But you're only supposed to work on cars for people who don't fix their own car. We cannot reimburse you for this claim.
Mechanic:	So you're saying I should not have fixed it myself?
Accountant:	You got it.
Mechanic:	Great, so I would be one of the folks who don't fix their own car. Then my job is that *I* have to fix it. Just what I did! And here's the claim.
Accountant:	You have a claim for your own car?

You can see this is going nowhere fast. Next, we will consider a somewhat more formal version of this nonsense dialog.

2 An Unlikely Set

The previous story comes in many variations, all of them related to potential problems with our notion of sets. Now we tell the same story again, however in the language of set theory.

A set may have other sets as members: the power set of a set is an example. So let us now concentrate on those sets which have sets as members. One such set would be the *set of all sets*. This set needs to have itself as a member... and is beginning to sound strange.

But it gets worse. The set of all sets has itself as a member, and there may be more sets with that property. Other sets of sets may not have themselves as a member. Call the set of all those sets $\mathcal{U}$. Question: does $\mathcal{U}$ have itself as a member?

There are two possibilities: a) it does, b) it does not. Let's explore a). This states that $\mathcal{U}$ has itself as a member. But since $\mathcal{U}$ only contains sets which do *not* have themselves as a member, option a) is ruled out right away. That leaves b), meaning that $\mathcal{U}$ does not have itself as a member. Hence $\mathcal{U}$ must be a member of itself, which is option a)!

We have arrived at a logical deadlock: the definition of $\mathcal{U}$ does not make sense, or it is self-contradictory.

3 Hilbert's Hotel

The concept of infinite sets can be puzzling. D. Hilbert provided the amusing concept of an infinite hotel, i. e., a hotel whose rooms are *all* the natural numbers. Suppose this hotel is full, and a new guest arrives. How can he be given a room when the hotel is full? Simply move all guests up one room: this will free room 0, and the new guest is accommodated. The procedure may of course be repeated. How about another infinite hotel being evacuated and all its guests are seeking a room in our fully booked hotel? No problem: move every guest to a room with twice the old room number, and all newcomers have a room!

4 $I\!R$ vs. $[0, 1]$

Another puzzling question is this: does the real line have more points on it than does the (unit) interval $[0, 1]$? Intuitively, one would think so. However, both point sets have the same number of points! This is in fact not hard to see at all by inspection of Fig. 1. There, the interval $[0, 1]$ is shaded gray. A semicircle is added having center $(1/2, 1/2)$ and radius $1/2$. A point in the interval $[0, 1]$ is then mapped to a point on the real line by the simple two-step construction shown in the figure. Note that the construction works equally well the other way around.

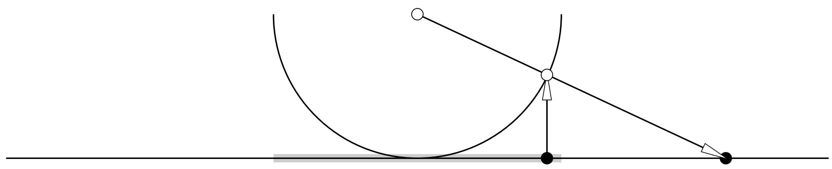

Figure 1: Relating a point in $[0, 1]$ to a point on the real line.

Since we have established a one-to-one relation between points on the real line and those in $[0, 1]$, we have shown that both sets have the same number of points!

5 $[0, 1]$ vs. $[0, 1]^2$

We have seen that the unit interval has as many points as does all of the real line. How about the unit square? That has one more dimension, and in that sense is far more populated with points than the unit interval. Surprisingly, no more points! There are several ways to show this, one involving space filling curves.

Here, we briefly present a different argument. Let x be a number in $[0, 1]$. It has a decimal expansion $0.d_1 d_2 d_3 ....$