2D Transformations
Introduction to Computer Graphics
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1 Introduction

When you see computer graphics images moving, spinning, or changing shape, you are seeing an implementation of 2D or 3D transformations. Let's focus on 2D here; once you understand 2D, 3D is simple!

2D transformations can be posed as the following problem.

Given: $\mathbf{v}$ in the $[\mathbf{e}_1, \mathbf{e}_2]$-coordinate system, where $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$.

Find: $\hat{\mathbf{v}}$ in the $[\mathbf{a}_1, \mathbf{a}_2]$-coordinate system, where $\hat{\mathbf{v}} = \hat{v}_1\mathbf{a}_1 + \hat{v}_2\mathbf{a}_2$.

The geometry of this problem is illustrated in Figure 1. Another name for a 2D transformation is a *2D linear map.* The purpose of this write-up is to define 2D linear maps.

Notation: the elements of $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$.

To define the 2D transformation that takes $\mathbf{v}$ to $\hat{\mathbf{v}}$, notice that in addition we want:

$$\mathbf{e}_1 \to \mathbf{a}_1 \quad \text{and} \quad \mathbf{e}_2 \to \mathbf{a}_2. \tag{1}$$

This mapping will be defined as a 2×2 matrix, and it is easy to see that

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A short-hand way to write these maps:

$$\mathbf{a}_1 = [\mathbf{a}_1, \mathbf{a}_2]\mathbf{e}_1 \quad \text{and} \quad \mathbf{a}_2 = [\mathbf{a}_1, \mathbf{a}_2]\mathbf{e}_2.$$

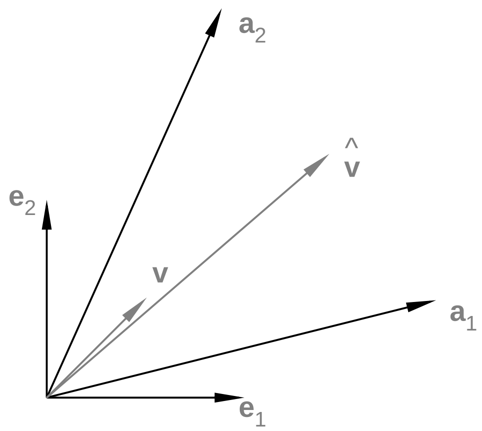

Figure 1: Elements defining a linear map: $[\mathbf{e}_1, \mathbf{e}_2] \rightarrow [\mathbf{a}_1, \mathbf{a}_2]$.

We'll write matrices in terms of their column vectors frequently during this course.

Therefore, the vector $\mathbf{v}$ is mapped to $\hat{\mathbf{v}}$ by

$$\begin{aligned} \hat{\mathbf{v}} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{v} \\ &= [\mathbf{a}_1, \mathbf{a}_2]\mathbf{v} \\ &= A\mathbf{v} \end{aligned} \tag{2}$$

The matrix A is called a *linear map*. Linear maps operate on vectors. (However, you will see them applied to points which are bound to a coordinate origin.)

2 Determinants and Area

Vectors $[\mathbf{e}_1, \mathbf{e}_2]$ define a parallelogram (unit square) with area one. Vectors $[\mathbf{a}_1, \mathbf{a}_2]$ define a parallelogram with area

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

where $|A|$ is called the *determinant* of matrix A. In other words, A maps the unit square to a parallelogram of area $|A|$.

- $|A| = 1$: no area change
- $0 < |A| < 1$: shrink
- $|A| > 1$: expand
- $|A| < 0$: change orientation

Handy relation: $|AB| = |A||B|$.

3 2D Linear Maps

All linear maps are composed of the following basic maps: scale, rotate, shear, projection.

1. Scale uniformly: $\hat{\mathbf{v}} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mathbf{v}$. See Figure 2. $|A| = s^2$.
2. Scale non-uniformly: $\hat{\mathbf{v}} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \mathbf{v}$. See Figure 3. $|A| = s_1 s_2$.

Reflections:

Here are some examples of reflection matrices. A reflection will have $|A| = -1$

1. Reflections about the $\mathbf{e}_1$-axis: $\hat{\mathbf{v}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}$. See Figure 4. $|A| = -1$.

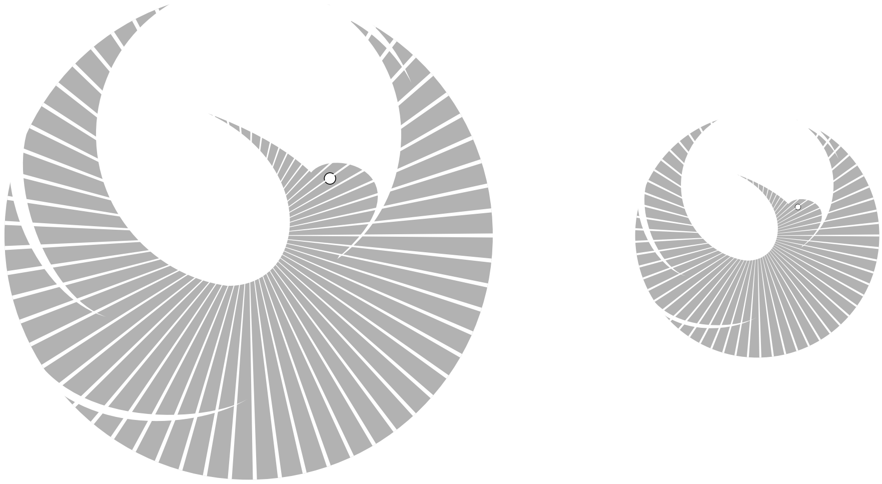

Figure 2: Uniform scale: $\hat{\mathbf{v}} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{v}$.

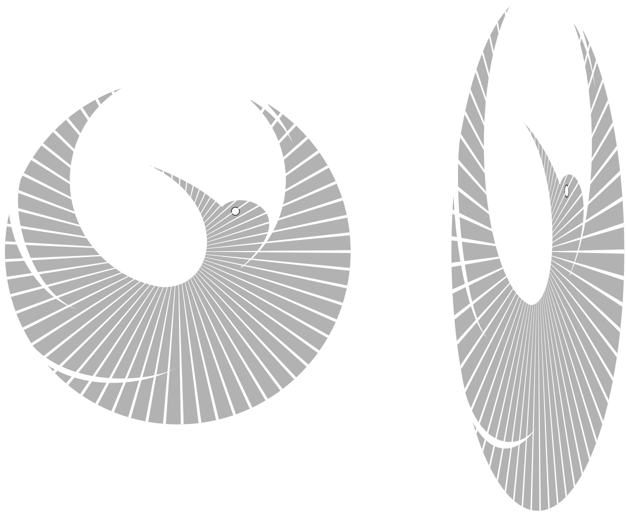

Figure 3: Non-uniform scale: $\hat{\mathbf{v}} = \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix} \mathbf{v}$.

Figure 4: Reflection about the $\mathbf{e}_1$-axis $\hat{\mathbf{v}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}$.

2. Reflection about the line $x_1 = x_2$: $\hat{\mathbf{v}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}$. See Figure 5. $|A| = -1$.

Rotation:

A rotation of α° is achieved by

$$\hat{\mathbf{v}} = \begin{bmatrix} \cos(\alpha^\circ) & -\sin(\alpha^\circ) \\ \sin(\alpha^\circ) & \cos(\alpha^\circ) \end{bmatrix} \mathbf{v}.$$

Rotations of positive degrees ($\alpha > 0$) result in a counterclockwise rotation. A rotation does not change areas, that is, $|A| = \cos^2(\alpha^\circ) + \sin^2(\alpha^\circ) = 1$. Additionally, a rotation is a *rigid body motion* because the shape of an object is not changed by a rotation.

Rotate 45°: $\hat{\mathbf{v}} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} \mathbf{v}$. See Figure 6.

Shear:

We can shear in either or both axes.

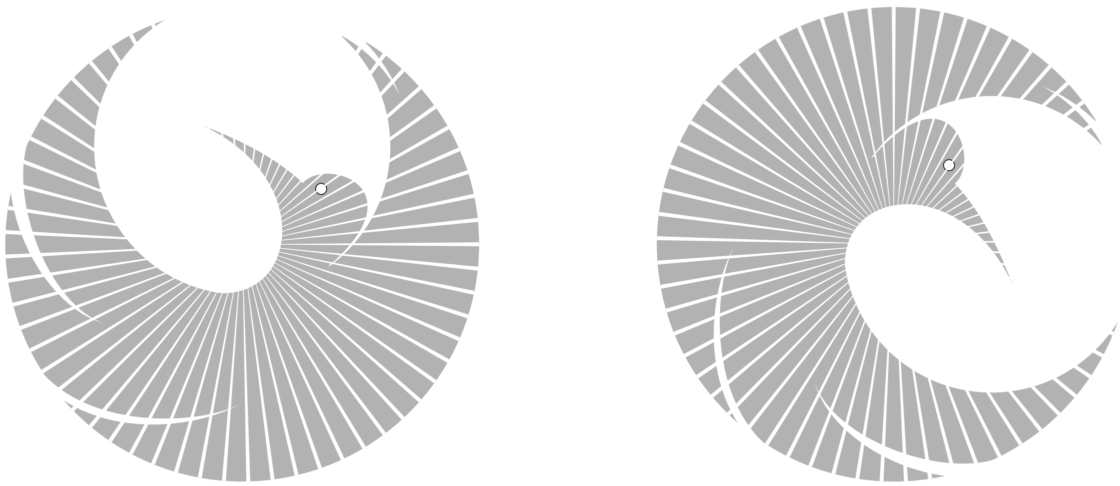

Figure 5: Reflection about the line $x_1 = x_2$: $\hat{\mathbf{v}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v}$.

Shear in the $\mathbf{e}_1$ direction $\hat{\mathbf{v}} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \mathbf{v}$. See Figure 7. $|A| = 1$. (Notice that the parallelogram maintains a base and height of one!)

Projections:

All vectors are projected onto a projection line. The angle of incidence with the projection line characterizes the projection, as illustrated in Figure 8, and enumerate below.

1. Parallel: All vectors have the same angle of incidence with the projection line.

   (a) Orthographic - angle of incidence is ninety-degrees to the projection line.

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

   (b) Oblique - arbitrary angle to the projection line

   $$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

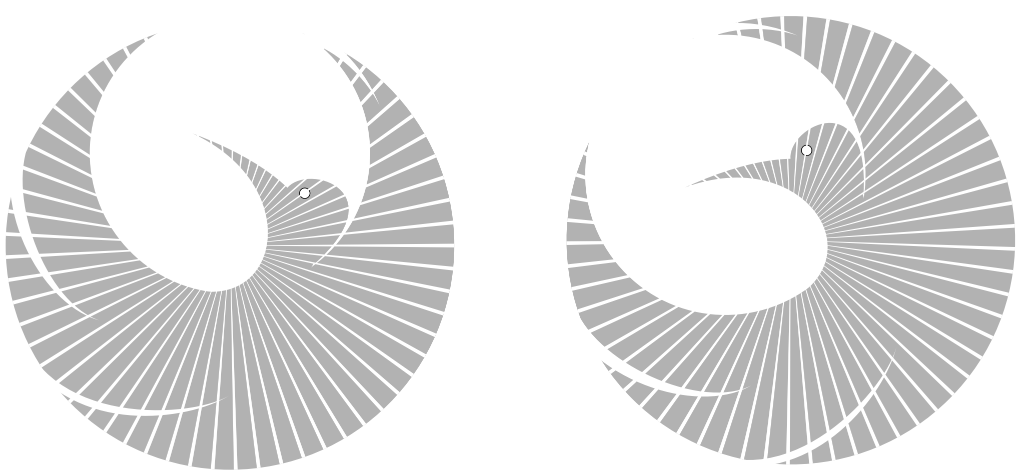

Figure 6: Rotate 45°: $\hat{\mathbf{v}} = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix} \mathbf{v}$.

2. General projections do not necessarily project all vectors with the same angle of incidence with the projection line. The general form:

$$A = \begin{bmatrix} \mathbf{a}_1 & c\mathbf{a}_1 \end{bmatrix}.$$

   The columns are *linearly dependent*. The number of linearly independent columns is the *rank* of a matrix. Here: rank equals one.

The determinant of a projection matrix: $|A| = 0$ since our unit square is squished to a line.

4 Properties of Linear Maps

A linear map, $\hat{\mathbf{v}} = A\mathbf{v}$,

- Maps vectors $\rightarrow$ vectors
- Preserves linear combinations of vectors. Suppose we have

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$$

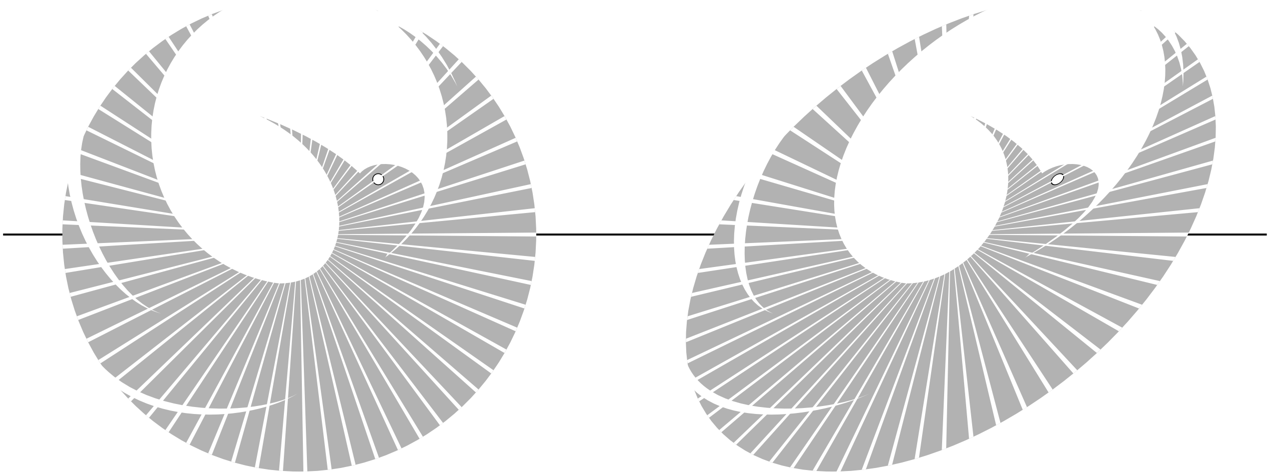

Figure 7: Shear in the $\mathbf{e}_1$ direction $\hat{\mathbf{v}} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \mathbf{v}$.

Apply the linear map:

$$\hat{\mathbf{v}} = A\mathbf{v} \qquad \hat{\mathbf{u}} = A\mathbf{u} \qquad \hat{\mathbf{w}} = A\mathbf{w}$$

And the relationship still holds:

$$\hat{\mathbf{w}} = \alpha\hat{\mathbf{u}} + \beta\hat{\mathbf{v}}$$

5 Matrix Multiplication

Matrix operations are not commutative: $AB \neq BA$ in general. For example, rotate$\times$reflect $\neq$ reflect$\times$rotate. See Figure 9.

However, rotations *in 2D* do commute: $R_\alpha R_\beta = R_\beta R_\alpha$. Additionally, $R_\alpha R_\beta = R_{\alpha+\beta}$.

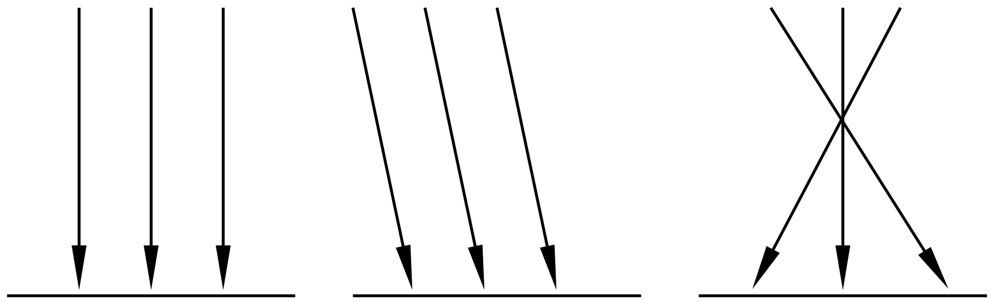

Figure 8: Projections are characterized by their angle of incidence with the projection line. Left: orthographic, Middle: oblique, Right: general.

6 Affine Maps

Modeling transformations include translations. This is where affine maps come in, and affine maps give us a tools for applying transformations to points.

Suppose we have a point $\mathbf{x}$ in the $[\mathbf{e}_1, \mathbf{e}_2]$-system, and we would like to find the corresponding point $\hat{\mathbf{x}}$ in a system defined by a point $\mathbf{p}$ and $[\mathbf{a}_1, \mathbf{a}_2]$. This is illustrated in Figure 10.

We need to apply a linear map to the vector $(\mathbf{x}-\mathbf{o})$ to find the corresponding vector in the $[\mathbf{a}_1, \mathbf{a}_2]$ system, and then translate the geometry by $\mathbf{p}$, or in other words,

$$\begin{aligned}\hat{\mathbf{x}} &= \mathbf{p} + A(\mathbf{x} - \mathbf{o}) \\ &= \mathbf{p} + A\mathbf{x}\end{aligned}$$

Thus an affine map is a translation plus a linear map. As mentioned in Section 1, it can appear that we apply a linear map to a point, but actually we are applying the linear map to the vector formed by the point and the

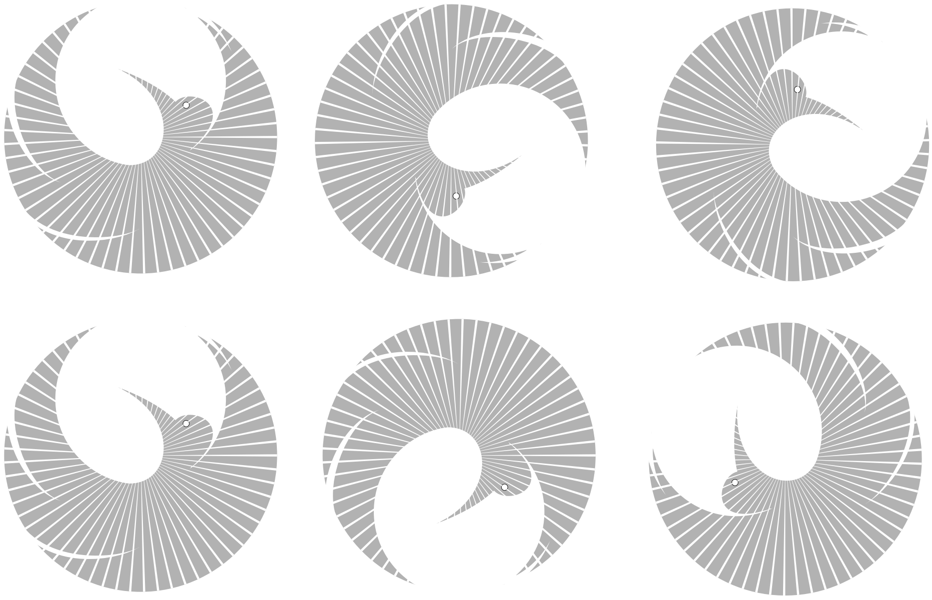

Figure 9: Matrix operations are not commutative. Top right: rotate($-120°$) $\times$ reflect(about x). This is not the same as the bottom right: reflect(about x) $\times$ rotate($-120°$).

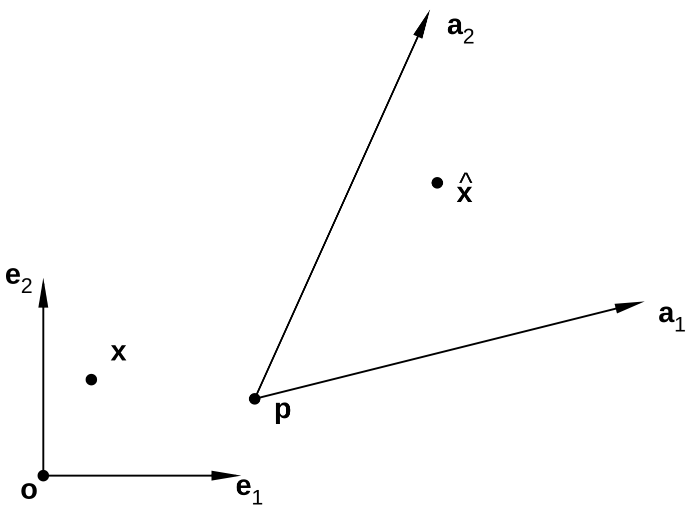

Figure 10: Affine map: defines the transformation of a point $\mathbf{x}$ in the $[\mathbf{e}_1, \mathbf{e}_2]$ system to a point $\hat{\mathbf{x}}$ in the system defined by $\mathbf{p}$ and $[\mathbf{a}_1, \mathbf{a}_2]$.

origin.

Affine maps have the following properties.

1. A point is mapped to a point.
2. The ratio of three collinear points is preserved. For example, suppose points $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{x}$ are collinear and $\text{ratio}(\mathbf{r}, \mathbf{x}, \mathbf{s}) = \alpha : \beta$. Apply an affine map to these points, resulting in points $\hat{\mathbf{r}}$, $\hat{\mathbf{s}}$, and $\hat{\mathbf{x}}$, then these points are collinear and $\text{ratio}(\hat{\mathbf{r}}, \hat{\mathbf{x}}, \hat{\mathbf{s}}) = \alpha : \beta$. See Figure 11.
3. Parallel lines are mapped to parallel lines.

Translations (along with rotations) are *rigid body motions* – they transform the geometry without deforming it.

Exercise: As practice in constructing affine maps, try creating the affine map (by defining A and $\mathbf{p}$) that takes the $\mathbf{o}, [\mathbf{e}_1, \mathbf{e}_2]$ local coordinates to the global coordinates defined by the target box $\mathbf{g}_{min}, \mathbf{g}_{max}$.

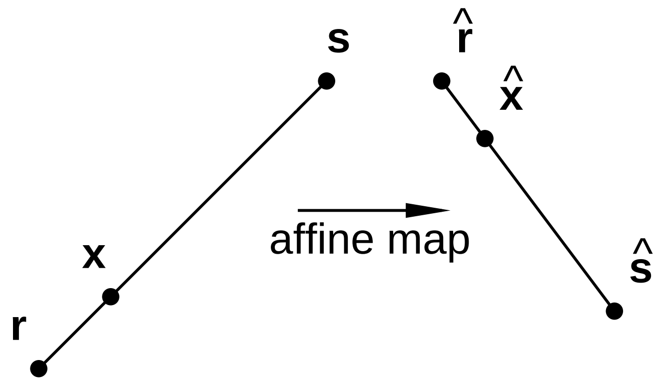

Figure 11: Affine maps preserve the ratio of three collinear points.

Problem: Given points $\mathbf{r}$ and $\mathbf{x}$, rotate $\mathbf{x}$ d degrees about $\mathbf{r}$. This geometry is illustrated in Figure 12.

This is a problem that arises frequently in computer graphics. A constructive approach is necessary as $\mathbf{a}_1$ and $\mathbf{a}_2$ are not given directly. One way to think about it: Translate $\mathbf{r}$ and $\mathbf{x}$ to the origin, rotate, and then translate the geometry back to its original position, or in other words,

$$\begin{aligned}\hat{\mathbf{x}} &= \mathbf{r} + A(\mathbf{x} - \mathbf{r}) \\ &= \mathbf{r} + \begin{bmatrix} \cos(d^\circ) & -\sin(d^\circ) \\ \sin(d^\circ) & \cos(d^\circ) \end{bmatrix} (\mathbf{x} - \mathbf{r}).\end{aligned}$$

Figures extracted from **The Geometry Toolbox for Graphics and Modeling** Gerald Farin and Dianne Hansford AK Peters, 2nd edition, 2004

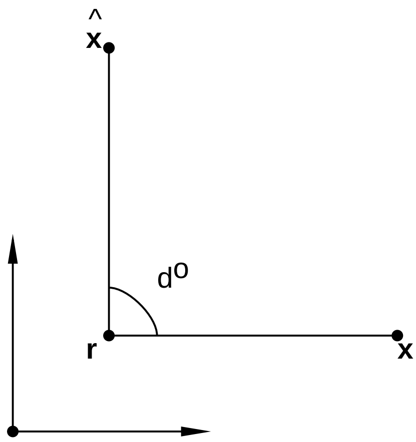

Figure 12: Problem: Rotate point $\mathbf{x}$ d° about the point $\mathbf{r}$, resulting in point $\hat{\mathbf{x}}$. Here: $d = 90^\circ$.