1 A Point and Many Planes

We consider a point $\mathbf{p}$ and a plane in implicit form

$$\mathbf{n} \cdot \mathbf{x} + c = 0.$$

The distance of $\mathbf{p}$ to the plane is given by

$$d = \mathbf{n} \cdot \mathbf{p} + c.$$

We simplify this expression by using homogenous coordinates

$$\underline{\mathbf{n}} = \begin{bmatrix} l \\ m \\ n \\ c \end{bmatrix} \quad \text{and} \quad \underline{\mathbf{p}} = \begin{bmatrix} r \\ p \\ q \\ 1 \end{bmatrix}.$$

Now the distance becomes:

$$d = \underline{\mathbf{n}}^{\mathrm{T}} \underline{\mathbf{p}}$$

and thus the square distance is given by

$$d^2 = (\underline{\mathbf{n}}^{\mathrm{T}} \underline{\mathbf{p}})^2.$$

This may be rewritten as

$$d^2 = (\underline{\mathbf{n}}^{\mathrm{T}} \underline{\mathbf{p}})(\underline{\mathbf{n}}^{\mathrm{T}} \underline{\mathbf{p}}) = (\underline{\mathbf{p}}^{\mathrm{T}} \underline{\mathbf{n}})(\underline{\mathbf{n}}^{\mathrm{T}} \underline{\mathbf{p}}) = \underline{\mathbf{p}}^{\mathrm{T}} [\underline{\mathbf{n}}\underline{\mathbf{n}}^{\mathrm{T}}] \underline{\mathbf{p}}.$$

The term $\underline{\mathbf{n}}\underline{\mathbf{n}}^{\mathrm{T}}$ is a symmetric 4x4 matrix, denoted by Q:

$$Q = \begin{bmatrix} l^2 & lm & ln & lc \\ lm & m^2 & mn & mc \\ ln & mn & n^2 & nc \\ lc & mc & nc & c^2 \end{bmatrix}.$$

Summarizing, the square distance is given by

$$d^2 = \underline{\mathbf{p}}^{\mathrm{T}} Q \underline{\mathbf{p}}$$

Now suppose we are given several planes (again, in homogeneous form):

$$\underline{\mathbf{n}}_i^{\mathrm{T}} \underline{\mathbf{x}} = 0.$$

In this form, the square distance of $\mathbf{p}$ to the i^{th} plane is given by

$$d_i^2 = \underline{\mathbf{p}}^{\mathrm{T}} Q_i \underline{\mathbf{p}}.$$

The *squared distance*D of $\mathbf{p}$ to all planes is given by

$$D = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \underline{\mathbf{p}}^{\mathrm{T}} Q_i \underline{\mathbf{p}}$$

If we define $M = Q_1 + \ldots + Q_n$, we have

$$D = \underline{\mathbf{p}}^{\mathrm{T}} M \underline{\mathbf{p}}.$$

2 An Optimization Problem

We are now equipped to solve the following optimization problem: Given is a line $\mathbf{p}(t) = \mathbf{a} + t\mathbf{v}$ and a set of planes given by their homogeneous representations $\underline{\mathbf{n}}_i$. For which value of t is the point $\mathbf{p}(t)$ on the line closest to all planes in the sense that the squared distance of $\mathbf{p}(t)$ to all planes is minimal? In other words, which t minimizes the function $D(t)$ given by

$$D(t) = \underline{\mathbf{p}}^{\mathrm{T}} M \underline{\mathbf{p}} = [\underline{\mathbf{a}} + t\underline{\mathbf{v}}]^{\mathrm{T}} M [\underline{\mathbf{a}} + t\underline{\mathbf{v}}]?$$

Here, $\underline{\mathbf{v}}$ is the homogeneous form of $\mathbf{v}$, i.e., $\underline{\mathbf{v}}^{\mathrm{T}} = [\mathbf{v}^{\mathrm{T}}, 0]$. We rewrite $D(t)$ as

$$D(t) = \underline{\mathbf{a}}^{\mathrm{T}} M \underline{\mathbf{a}} + 2t\underline{\mathbf{a}}^{\mathrm{T}} M \underline{\mathbf{v}} + t^2 \underline{\mathbf{v}}^{\mathrm{T}} M \underline{\mathbf{v}}.$$

This is a quadratic function in t, such as shown in Fig. **??**. It is nonnegative by the definition of D (being a square distance), hence D has a minimum for some t.[1] We calculate t by setting the derivative of D with respect to t equal to zero. This leads to

$$t = -\frac{\underline{\mathbf{a}}^{\mathrm{T}} M \underline{\mathbf{v}}}{\underline{\mathbf{v}}^{\mathrm{T}} M \underline{\mathbf{v}}}.$$

The situation is similar if somewhat more complicated if we do not restrict $\mathbf{p}$ to be on a given line. Then it may be written as

$$\mathbf{p} = \mathbf{e} + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

with $\mathbf{e} = [0, 0, 0]^{\mathrm{T}}$. Our function D becomes

$$D(x, y, z) = [\underline{\mathbf{e}} + x\underline{\mathbf{e}}_1 + y\underline{\mathbf{e}}_2 + z\underline{\mathbf{e}}_3]^{\mathrm{T}} M [\underline{\mathbf{e}} + x\underline{\mathbf{e}}_1 + y\underline{\mathbf{e}}_2 + z\underline{\mathbf{e}}_3].$$

Again, this is a quadratic function, now in three variables x, y, z. If it has a minimum somewhere, then its partials have to vanish:

$$\frac{\partial D}{\partial x} = 0, \quad \frac{\partial D}{\partial y} = 0, \quad \frac{\partial D}{\partial z} = 0.$$

We find (using the product rule):

$$\begin{aligned}
\frac{\partial D}{\partial x} = 0 \quad &= [\underline{\mathbf{e}} + x\underline{\mathbf{e}}_1 + y\underline{\mathbf{e}}_2 + z\underline{\mathbf{e}}_3]^{\mathrm{T}} M \underline{\mathbf{e}}_1 \\
\frac{\partial D}{\partial y} = 0 \quad &= [\underline{\mathbf{e}} + x\underline{\mathbf{e}}_1 + y\underline{\mathbf{e}}_2 + z\underline{\mathbf{e}}_3]^{\mathrm{T}} M \underline{\mathbf{e}}_2 \\
\frac{\partial D}{\partial z} = 0 \quad &= [\underline{\mathbf{e}} + x\underline{\mathbf{e}}_1 + y\underline{\mathbf{e}}_2 + z\underline{\mathbf{e}}_3]^{\mathrm{T}} M \underline{\mathbf{e}}_3.
\end{aligned}$$

These are three equations in the three unknowns x, y, z. If all planes are parallel, there will not be a solution.

[1] We should note that in the case of all planes being parallel to $\mathbf{v}$, no solution exists.