Quaternions

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1 Motivation

For animation, objects have to be moved through space. If the object's orientation changes during such a motion, a rotation is invariably involved. A typical problem is that the object is only defined at several key positions, and the intermediate ones have to be computed. In other words, we have to be able to interpolate between rotations. This is not trivial: if R_1 and R_2 are two rotation matrices, then the blend

$$Q = (1-t)R_1 + tR_2$$

will, in general, not be a rotation matrix since we cannot guarantee that $Q^{\mathrm{T}}Q = I$ as is required for rotation matrices. A different entity, called a quaternion, may also be used to describe rotations, but in addition is capable of handling the interpolation problem.

2 Introduction

Let R be rotation matrix, such that

$$R^T R = I.$$

First we show that R has an eigenvalue 1, i.e., there is a unit vector $\mathbf{u}$, corresponding to this eigenvalue, such that

$$A\mathbf{u} = \mathbf{u}.$$

Assume λ is the eigenvector corresponding to $\mathbf{u}$. Then

$$(\lambda\mathbf{u})^{\mathrm{T}}(\lambda\mathbf{u}) = \lambda^2\mathbf{u}^{\mathrm{T}}\mathbf{u} = \mathbf{u}^{\mathrm{T}}R^{\mathrm{T}}R\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{u} = 1,$$

hence $\lambda^2 = \pm 1$. To show that in fact $+1$ is an eigenvector requires some more work.

Thus in order to find the axis of a general rotation matrix, all we have to do is to find its eigenvector corresponding to the eigenvalue +1. The angle Φ by which $\mathbf{v}$ is rotated around $\mathbf{u}$ to yield $\mathbf{v}'$ can be computed from

$$\mathrm{trace}(R) = 1 + 2\cos\Phi.$$

Example Let a rotation matrix be given by

$$R = \begin{bmatrix} \tau^2 & \tau^2 - \tau & \tau^2 + \tau \\ \tau^2 + \tau & \tau^2 & \tau^2 - \tau \\ \tau^2 - \tau & \tau^2 + \tau & \tau^2 \end{bmatrix}$$

with $\tau = 1/\sqrt{3}$. The matrix has an eigenvector $\mathbf{u} = [\tau, \tau, \tau]^{\mathrm{T}}$ corresponding to the eigenvalue 1:

$$\begin{bmatrix} \tau^2 & \tau^2 - \tau & \tau^2 + \tau \\ \tau^2 + \tau & \tau^2 & \tau^2 - \tau \\ \tau^2 - \tau & \tau^2 + \tau & \tau^2 \end{bmatrix} \begin{bmatrix} \tau \\ \tau \\ \tau \end{bmatrix} = \begin{bmatrix} \tau^3 + \tau^3 - \tau^2 + \tau^3 + \tau^2 \\ \tau^3 + \tau^2 + \tau^3 + \tau^3 - \tau^2 \\ \tau^3 - \tau^2 + \tau^3 + \tau^2 + \tau^3 \end{bmatrix} = \begin{bmatrix} \tau \\ \tau \\ \tau \end{bmatrix}$$

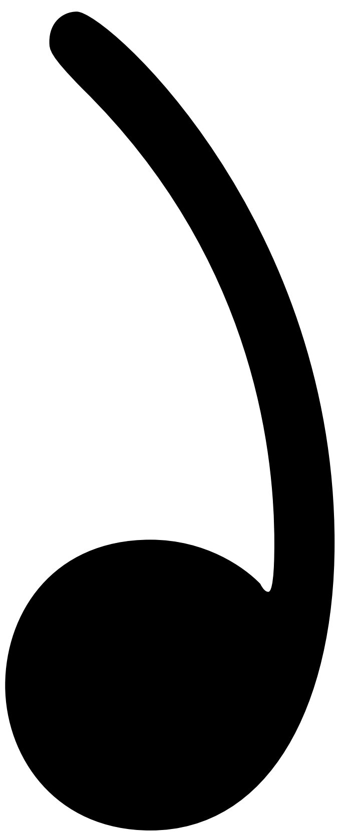
We have $\mathrm{trace}(R) = 3\tau^2 = 1$. Thus $\cos\Phi = 0$ and thus $\Phi = 90$.

If $\mathbf{v}' = R\mathbf{v}$, then $\mathbf{v}'$ is in the plane which contains $\mathbf{v}$ and has $\mathbf{u}$ as its normal. To see this, note that $(\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{u}$ is the perpendicular projection of $\mathbf{v}$ onto $\mathbf{u}$, marked by a solid circle in Figure 1. If we can show that the projection of $\mathbf{v}'$ onto $\mathbf{u}$ is the same vector, our claim is proved. That projection is given by $\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{u}$. But

$$(\mathbf{u}^{\mathrm{T}}\mathbf{v}')\mathbf{u} = (\mathbf{u}^{\mathrm{T}}R^{\mathrm{T}}R\mathbf{v})\mathbf{u} = (\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{u},$$

thus proving our claim.

Figure 1: The vector $\mathbf{v}$ is rotated into $\mathbf{v}'$ around axis $\mathbf{u}$.



Conversely, we may ask how to compute the rotation matrix R if we are given the axis vector $\mathbf{u}$ and the rotation angle Φ. This matrix turns out to be

$$\begin{bmatrix} u_1^2 + c(1-u_1^2) & (1-c)u_1u_2 - su_3 & (1-c)u_1u_3 + su_2 \\ (1-c)u_1u_2 + su_3 & u_2^2 + c(1-u_2^2) & (1-c)u_2u_3 - su_1 \\ (1-c)u_1u_3 - su_2 & (1-c)u_2u_3 + su_1 & u_3^2 + c(1-u_3^2) \end{bmatrix}, \tag{1}$$

where $c = \cos\Phi, s = \sin\Phi$. For a sanity check, verify that $\text{trace}(R) = 1 + 2c$!

The essential information defining a rotation is given by $\mathbf{u}$ and Φ. This information is collected in the *quaternion*

$$\hat{\mathbf{u}} = \begin{bmatrix} s\mathbf{u} \\ c \end{bmatrix} \tag{2}$$

Clearly we can rebuild the matrix R from this information. But instead of working with matrices, some applications related to animation are formulated much more elegantly using quaternions.

3 Definitions

Since quaternions were originally invented as a generalization of complex numbers, we now turn to a formulation which is more general than the above. Now, a quaternion is defined as

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{x} \\ w \end{bmatrix}$$

where

$$\mathbf{x} = ix + jy + kz$$

Note that on one hand, we write $\mathbf{x}$ as a vector, on the other hand as a generalized complex number. It's not a totally bad idea to think of i, j, k as the unit direction vectors of $\mathbb{E}^3$, but they are better thought of as generalized complex numbers:

$$i^2 = j^2 = k^2 = -1; \quad ij = k, jk = i, ki = j.$$

We also have $ij = -ji$ etc.

The product $\hat{\mathbf{q}}_1\hat{\mathbf{q}}_2$ of two quaternions may again expressed as a quaternion. It is found by multiplying

$$(ix_1 + jy_1 + kz_1 + w_1)(ix_2 + jy_2 + kz_2 + w_2)$$

which gives

$$\hat{\mathbf{q}}_1\hat{\mathbf{q}}_2 = \begin{bmatrix} w_1x_2 + x_1w_2 + y_1z_2 - z_1y_2 \\ w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2 \\ w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2 \\ w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2 \end{bmatrix}.$$

It will be helpful to rewrite this in matrix form:

$$\hat{\mathbf{q}}_1\hat{\mathbf{q}}_2 = \begin{bmatrix} w_1 & -z_1 & y_1 & x_1 \\ z_1 & w_1 & -x_1 & y_1 \\ -y_1 & x_1 & w_1 & z_1 \\ -x_1 & -y_1 & -z_1 & w_1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{bmatrix}$$

This multiplication is clearly not commutative!

Other operations on quaternions are most easily understood by knowing they are equivalent to matrices. For example, addition is defined by

$$\hat{\mathbf{q}} + \hat{\mathbf{p}} = \begin{bmatrix} \mathbf{q}_v + \mathbf{p}_v \\ q_w + p_w \end{bmatrix}.$$

Another property is linearity:

$$\hat{\mathbf{q}}(\alpha\hat{\mathbf{p}}_1 + \beta\mathbf{p}_2) = \alpha\hat{\mathbf{q}}\hat{\mathbf{p}}_1 + \beta\hat{\mathbf{q}}\hat{\mathbf{p}}_2.$$

Since the i, j, k parts are imaginary numbers, we may define the complex conjugate $\hat{\mathbf{q}}^*$ of a quaternion $\hat{\mathbf{q}}$. First, recall the standard complex conjugate of a complex number:

$$(a + ib)^* = a - ib.$$

For quaternions, the same pattern holds:

$$\hat{\mathbf{q}}^* = \begin{bmatrix} -\mathbf{x} \\ w \end{bmatrix}.$$

The *norm* (length) of a complex number is given by $|a + ib|^2 = (a + ib)(a + ib)^* = (a + ib)(a - ib) = a^2 - aib + aib - i^2b = a^2 + b^2$. The same holds for quaternions:

$$\|\hat{\mathbf{q}}\|^2 = \hat{\mathbf{q}}\hat{\mathbf{q}}^*.$$

Note that $\hat{\mathbf{q}}\hat{\mathbf{q}}^*$ yields a real number because of the properties of the imaginary units in quaternions.

A *unit quaternion* is one with $\|\hat{\mathbf{q}}\|^2 = 1$ and ties in with the quaternions introduced in Section 3. It is thus of the form of (refeq:unitquat):

$$\hat{\mathbf{q}} = \begin{bmatrix} \sin\Phi\mathbf{u}_q \\ \cos\Phi \end{bmatrix} \tag{3}$$

We see that $\hat{\mathbf{q}}\hat{\mathbf{q}}^* = s^2(u_1^2 + u_2^2 + u_3^2) + c^2 = 1$ since we assumed that $\mathbf{u}_q$ is a unit vector. In what follows, we will only deal with unit quaternions.

Finally, the identity quaternion:

$$\hat{\mathbf{i}} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

We may also define the inverse $\hat{\mathbf{q}}^{-1}$ of $\hat{\mathbf{q}}$, defined by

$$\hat{\mathbf{q}}^{-1}\hat{\mathbf{q}} = \hat{\mathbf{q}}\hat{\mathbf{q}}^{-1} = 1.$$

It is given by

$$\hat{\mathbf{q}}^{-1} = \frac{\hat{\mathbf{q}}^*}{\|\hat{\mathbf{q}}\|^2},$$

and is verified by

$$\frac{\hat{\mathbf{q}}^{-1}\hat{\mathbf{q}}^*}{\|\hat{\mathbf{q}}\|^2} = 1.$$

4 Quaternions and Matrices

We already saw how to extract a quaternion from a matrix. A common situation is this: having performed calculation using quaternions, we will need to transform the result back into a matrix. Assuming we are dealing with unit quaternions (3), the corresponding matrix must correspond to (1), and is given by

$$R = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix}$$

Note that $\mathrm{trace}(R) = 3 - 4(x^2+y^2+z^2) = -1$. That does not make sense! more work here!

5 Interpolation using quaternions

Suppose $\hat{\mathbf{q}}_0$ and $\hat{\mathbf{q}}_1$ are two unit quaternions, such that at time $t = 0$ we assume orientation $\hat{\mathbf{q}}_0$ and at time $t = 1$, we assume $\hat{\mathbf{q}}_1$. What is a meaningful orientation at an arbirary time t? The answe is given by *spherical linear interpolation* or *slerp*:

$$\hat{\mathbf{q}}(t) = \frac{\sin((1-t)\Phi)}{\sin\Phi}\hat{\mathbf{q}}_0 + \frac{\sin((t\Phi)}{\sin\Phi}\hat{\mathbf{q}}_1.$$

where $\cos\Phi = \hat{\mathbf{q}}_0\dot{\hat{\mathbf{q}}}_1$.