Directional Derivatives

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The image of a straight line through two points (u_0, v_0) and (u_1, v_1) in the domain of a tensor product patch is a Bézier curve

$$\mathbf{x}(t) = \sum_{k=0}^{2n} \sum_{i+j=k} \frac{\binom{n}{i}\binom{n}{j}}{\binom{2n}{k}} B_k^{2n} \mathbf{b}[u_0^{<n-i>}, u_1^{<i>} \mid v_0^{<n-j>}, v_1^{<j>}].$$

Explicitly, the first two Bézier points of this curve are

$$\mathbf{b}_0 = \mathbf{b}[u_0^{<n>} \mid v_0^{<n>}] \tag{1}$$

$$\mathbf{b}_1 = \frac{1}{2}\mathbf{b}[u_0^{<n-1>}, u_1] \mid v_0^{<n>}] + \frac{1}{2}\mathbf{b}[u_0^{<n>}] \mid v_0^{<n-1>}, v_1]. \tag{2}$$

If $\mathbf{d} = \mathbf{u}_1 - \mathbf{u}_0$ in the domain is a unit vector, then the directional derivative of the surface at $\mathbf{u}_0$ equals the derivative of the curve at $t = 0$ and is given by

$$D_{\mathbf{d}}\mathbf{x}(\mathbf{u}_0) = 2n\Big(\frac{1}{2}\mathbf{b}[u_0^{<n-1>}, u_1] \mid v_0^{<n>}] + \frac{1}{2}\mathbf{b}[u_0^{<n>}] \mid v_0^{<n-1>}, v_1] - \mathbf{b}[u_0^{<n>} \mid v_0^{<n>}]\Big).$$

The geometry behind this is illustrated in Figure 1. the points marked by hollow squares correspond to $\mathbf{b}[u_0^{<n-1>}, i] \mid \mathbf{b}[v_0^{<n-1>,j}]; 0 \le i, j \le 1$. The two solid circles correspond to $\mathbf{b}[u_0^{<n>}] \mid v_0^{<n>}]$ and to $\frac{1}{2}\mathbf{b}[u_0^{<n-1>}, u_1] \mid v_0^{<n>}] + \frac{1}{2}\mathbf{b}[u_0^{<n>}] \mid v_0^{<n-1>}, v_1]$.

Let us now consider the special case that $v_0 = v_1 = v$, i.e., $\mathbf{d} = [1, 0]^{\mathrm{T}}$. Let us also replace $\mathbf{u}_0$ by a generic $\mathbf{u}$. Then

$$D_{\mathbf{d}}\mathbf{x}(\mathbf{u}) = \frac{\partial \mathbf{x}(\mathbf{u})}{\partial u}.$$

Now we have

$$D_{\mathbf{d}}\mathbf{x}(\mathbf{u}) = 2n\Big(\frac{1}{2}\mathbf{b}[u^{<n-1>}, u + \vec{1}] \mid v^{<n>}] + \frac{1}{2}\mathbf{b}[u^{<n>}] \mid v^{<n>}] - \mathbf{b}[u^{<n>} \mid v^{<n>}]\Big)$$

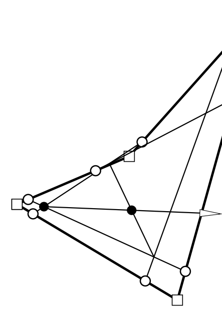

Figure 1: Directional derivatives and blossoms.

which may be simplified to

$$D_{\mathbf{d}}\mathbf{x}(\mathbf{u}) = n\big(\mathbf{b}[u^{<n-1>}, u+\vec{1}] \mid v^{<n>}] - \mathbf{b}[u^{<n>} \mid v^{<n>}]\big) \tag{3}$$
$$= n\mathbf{b}[u^{<n-1>}, \vec{1} \mid v^{<n>}]. \tag{4}$$

Figure 2 shows several directional derivatives.

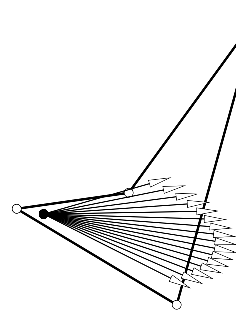

Figure 2: Several directional derivatives.