Smoothness of B-spline curves

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March 31, 2002

Consider a knot sequence u_0, u_1, u_3, u_4 and B-spline control points $\mathbf{d}[u_0, u_1], \mathbf{d}[u_1, u_3], \mathbf{d}[u_3, u_4]$. What we really have here is a quadratic curve, written in B-spline form. Let us carry out one level of the de Boor algorithm for a parameter value $u_2 \in [u_1, u_3$. We obtain the following scheme:

$$\begin{array}{ll} \mathbf{d}[u_0, u_1] & \\ \mathbf{d}[u_1, u_3] & \mathbf{d}[u_1, u_2] \\ \mathbf{d}[u_3, u_4] & \mathbf{d}[u_2, u_3]. \end{array}$$

We may continue with a different argument $u \in [u_1, u_3]$, thus obtaining values $\mathbf{d}[u_2, u]$.

If we now break the relationship between the first and the second row, i.e., we assign arbitrary values to $\mathbf{d}[u_1, u_2], \mathbf{d}[u_2, u_3]$, we still have a well-defined linear blossom at u_2, namely $\mathbf{d}[u_2, u]$. In other words, if we consider the knot sequence u_0, u_1, u_2, u_3, u_4 and B-spline control points $\mathbf{d}[u_0, u_1], \mathbf{d}[u_1, u_2]\mathbf{d}[u_2, u_3], \mathbf{d}[u_3, u_4],$, then there is a well-defined linear blossom $\mathbf{d}[u_2, u]$ at u_2. We note that $\mathbf{d}[u_2, u]$ represents the tangent line at $\mathbf{d}[u_2, u_2]$.

We may evaluate this blossom for a vector argument $\mathbf{d}[u_2, \vec{1}]$, thus verifying that our quadratic B-spline curve has a continuous first derivative at u_2. But it does not have a second derivative at u_2: a second derivative is of the form $\mathbf{d}[\vec{1}, \vec{1}]$; however, we have already “used up” one argument u_2.

The above reasoning is easily generalized. Let a knot $u_i \in [u_{i-1}, u_{i+1}]$ be of multiplicity r. Then there is a blossom associated with u_i, namely $\mathbf{d}[u_i^{<r>}, v_1, \ldots, v_{n-r}]$ which may be interpreted as resulting from a blossom for the interval $[u_{i-1}, u_{i+1}]$, evaluated r times at u_i. This is a blossom with only $n - r$ variables $v_1, \ldots, v_{n-r}$. The curve $\mathbf{d}[u_i^{<r>}, u^{<n-r>}]$ is the *polar* at u_i. We may form derivatives up to $\mathbf{d}[u_i^{<r>}, \vec{1}^{<n-r>}]$, thus showing that a B-spline curve has $n - r$ continuous derivatives at a knot of multiplicity r.

A cubic example: Let the knot sequence be $u_0, u_1, u_2, u_3, u_3, u_4, u_5, u_6$ resulting in control points

$$\mathbf{d}[u_0, u_1, u_2], \mathbf{d}[u_1, u_2, u_3], \mathbf{d}[u_2, u_3, u_3], \mathbf{d}[u_3, u_3, u_4], \mathbf{d}[u_3, u_4, u_5], \mathbf{d}[u_4, u_5, u_6].$$

The blossom associated with u_3 is $\mathbf{d}[u_3, u_3, u]$. It is linear in u and thus a derivative $\mathbf{d}[u_3, u_3, \vec{1}]$ may be defined, but not a second derivative.