More on linear interpolation

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1 Vector arguments

Since linear interpolation is an affine map, we have the following:

$$\mathbf{l}((1-s)c+sd)=(1-s)\mathbf{l}(c)+s\mathbf{l}(d). \tag{1}$$

This simply states that a relationship of points in the domain is left unchanged for the corresponding points i the range.

We may take this one step further: if a vector is defined in the domain, then that definition also carries over to the range. If c and d are two points on the real line, the domain, they define a vector α by setting $\alpha = d - c$. The corresponding vector in the range is then defined as

$$\mathbf{l}(\alpha)=\mathbf{l}(d)-\mathbf{l}(c). \tag{2}$$

2 Vectors and blossoms

Menelaos' theorem is an assertion about the commmutativity of supplying two different (point) arguments to piecewise linear interpolation of a polygon consisting of three points. Instead of using two points in the real line as arguments, we may use a point and a vector. If t the point and α is the vector in the real line, our generalized Menelaos' theorem states

$$\mathbf{l}(t,\alpha)=\mathbf{l}(\alpha,t). \tag{3}$$

It is easy to check that both expressions equal $(t-1)\mathbf{p}_1+(1-2t)\mathbf{p}_2+t\mathbf{p}_3$.

3 Derivatives and blossoms: affine

First, an identity, closely resembling Leibniz' formula for derivatives. Let t be a point on the real line and let $\vec{v}$ be a vector in the associated 1D linear space. Then

$$\mathbf{b}[(t+\vec{v})^{<n>}] = \sum_{i=0}^{n} \binom{n}{i} \mathbf{b}[t^{<n-i>}, \vec{v}^{<i>}]. \tag{4}$$

This is an immediate consequence of the Leibniz formula (**??**).

The derivative of a curve $\mathbf{x}(t)$ is typically defined as

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \lim_{h\to 0} \frac{1}{h}[\mathbf{x}(t+h) - \mathbf{x}(t)].$$

We will be a little more precise and observe that t is a 1D point whereas h is a 1D vector. We thus denote it by $\vec{h}$ and obtain

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \lim_{\vec{h}\to \vec{0}} \frac{1}{|\vec{h}|}[\mathbf{x}(t+\vec{h}) - \mathbf{x}(t)].$$

Invoking (4), we have

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \lim_{\vec{h}\to \vec{0}} \frac{1}{|\vec{h}|}\Big[\sum_{i=0}^{n} \binom{n}{i} \mathbf{b}[t^{<n-i>}, \vec{h}^{<i>}] - \mathbf{b}[t^{<n>}]\Big]. \tag{5}$$

There are two terms $\mathbf{b}[t^{<n>}]$ which cancel. We expand the rest and factor in the term $|\vec{h}|$:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \lim_{\vec{h}\to \vec{0}} \Big(n\mathbf{b}[t^{<n-1>}, \frac{\vec{h}}{|\vec{h}|}] + \binom{n}{2} \mathbf{b}[t^{<n-2>}, \frac{\vec{h}}{|\vec{h}|}, \vec{h}] + \ldots \Big)$$

We observe that $\frac{\vec{h}}{|\vec{h}|} = \vec{1}$. Taking the the limit annihilates all other terms containing $\vec{h}$, and we thus have

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = n\mathbf{b}[t^{<n-1>}, \vec{1}]. \tag{6}$$

From now on, we use the expression $\dot{\mathbf{x}}(t)$ for the first derivative.

4 Derivatives and blossoms: projective

The above reasoning has to change if we study derivatives in projective space. Since there are no vectors, we replace the vector α by the fundamental point f, corresponding to $t = \infty$. We fix a parameter value t_0; this is where we want to compute the derivative. It is defined by

$$\dot{\mathbf{x}}(t_0) = \lim_{t \to 0} \frac{1}{h}[\mathbf{x}(t_0 + tf) - \mathbf{x}(t_0)].$$

Proceeding exactly as above, we obtain

$$\dot{\mathbf{x}}(t_0) = n\underline{\mathbf{b}}[t_0^{<n-1>}, f]. \tag{7}$$

Since f corresponds to $t = \infty$, we may also write

$$\dot{\mathbf{x}}(t_0) = n\underline{\mathbf{b}}[t_0^{<n-1>}, \infty]. \tag{8}$$