Smoothness of B-spline curves

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Let U and V be two subsequent intervals in a knot sequence. Define a blossom $\mathbf{d}^U$ over U with control vertices $\mathbf{d}^U[U_i^{n-1}]; i = 0, \ldots, n$ and also define a blossom $\mathbf{d}^V$ over V with control vertices $\mathbf{d}^V[V_i^{n-1}]; i = 0, \ldots, n$. (Recall that U_i^{n-1} is the i^{th} $n-$tuple of knots containing U.) Since U and V are adjacent, there is overlap in the control vertex definitions.

Example: let $n = 2$ and let the knot sequence be u_0, u_1, u_2, u_3, u_4. Set $U = [u_1, u_2]$ and $V = [u_2, u_3]$. Then the control points for $\mathbf{d}^U$ are $\mathbf{d}^U[u_0, u_1], \mathbf{d}^U[u_1, u_2], \mathbf{d}^U[u_2, u_3]$ and those for $\mathbf{d}^V$ are $\mathbf{d}^V[u_1, u_2], \mathbf{d}^V[u_2, u_3], \mathbf{d}^V[u_3, u_4]$. Note that the arguments of $\mathbf{d}^U[*, *]$ contain at least one of U's endpoints; similarly, the $\mathbf{d}^V[*, *]$ involve at least one of V's endpoints. We define control points to be equal if their blossom arguments agree; for those control points, we drop the superscript.[1] Thus we have control points

$$\mathbf{d}^U[u_0, u_1], \mathbf{d}[u_1, u_2], \mathbf{d}[u_2, u_3], \mathbf{d}^V[u_3, u_4].$$

These are the control points of the composite blossom $\mathbf{d}$; when the context is clear, we will drop all superscripts. Figure 1 illustrates.

For the general case, we again identify control points with the same argument sequence and drop the superscript for those vertices: if $U_i^{n-1} = V_j^{n-1}$, then we set

$$\mathbf{d}^U[U_i^{n-1}] = \mathbf{d}^V[V_j^{n-1}] = \mathbf{d}[U_i^{n-1}].$$

We will now show that the composite blossom $\mathbf{d}$ defines a continuous curve. For general arguments t, we will have $\mathbf{d}^U[t^{<n>}] \neq \mathbf{d}^V[t^{<n>}]$. But if s is the knot common to both U and V, we will show

$$\mathbf{d}^U[s^{<n>}] = \mathbf{d}^V[s^{<n>}].$$

Again, we illustrate the quadratic example (where we have $s = u_2$). The two blossom

[1] One does not have to adopt this definition, but it is what makes B-splines work.

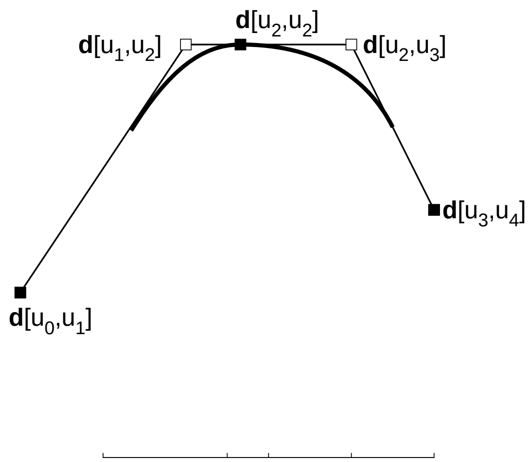

Figure 1: Composite blossoms: the case of two intervals $U = [u_1, u_2]$ and $V = [u_2, u_3]$
.

evaluations

$$\begin{array}{lll} & u_2 & u_2 \\ & \downarrow & \downarrow \\ \mathbf{d}^U[u_0, u_1] & & \\ \mathbf{d}^U[u_1, u_2] & \mathbf{d}^U[u_1, u_2] & \\ \mathbf{d}^U[u_2, u_3] & \mathbf{d}^U[u_2, u_2] & \mathbf{d}^U[u_2, u_2] \end{array}$$

and

$$\begin{array}{lll} & u_2 & u_2 \\ & \downarrow & \downarrow \\ \mathbf{d}^V[u_1, u_2] & & \\ \mathbf{d}^V[u_2, u_3] & \mathbf{d}^V[u_2, u_2] & \\ \mathbf{d}^V[u_3, u_4] & \mathbf{d}^V[u_2, u_3] & \mathbf{d}^V[u_2, u_2] \end{array}$$

may be combined into one scheme:

$$\begin{array}{ccc} & u_2 & u_2 \\ & \downarrow & \downarrow \\ \mathbf{d}^U[u_0, u_1] & & \\ \mathbf{d}[u_1, u_2] & \mathbf{d}[u_1, u_2] & \\ \mathbf{d}[u_2, u_3] & \mathbf{d}[u_2, u_2] & \mathbf{d}[u_2, u_2] \\ \mathbf{d}^V[u_3, u_4] & \mathbf{d}[u_2, u_3] & \mathbf{d}[u_2, u_2] \end{array}$$

We have omitted the superscripts where the $U-$ and $V-$ blossoms yield identical values.

In the general case, the last two entries $\mathbf{d}^U[s^{<n>}]$ and $\mathbf{d}^V[s^{<n>}]$ are obtained from three preceding values $\mathbf{d}^U[*, s^{<n-1>}], \mathbf{d}[s^{<n>}], \mathbf{d}^V[*, s^{<n-1>}]$. We need to perform two linear interpolations at s, thus obtaining the value $\mathbf{d}[s^{<n>}]$ twice.

In much the same way, we can show that the composite blossom generates a C^{n-1} curve, in other words:[2]

$$\mathbf{d}^U[s^r, e^{<n-r>}] = \mathbf{d}^V[s^r, e^{<n-r>}]; r = 1, \ldots, n-1.$$

The quadratic case first: we evaluate with respect to e, then with respect to u_2:

$$\begin{array}{ccc} & e & u_2 \\ & \downarrow & \downarrow \\ \mathbf{d}^U[u_0,u_1] & & \\ \mathbf{d}[u_1,u_2] & \mathbf{d}^U[e,u_1] & \\ \mathbf{d}[u_2,u_3] & \mathbf{d}[e,u_2] & \mathbf{d}[e,u_2] \\ \mathbf{d}^V[u_3,u_4] & \mathbf{d}^V[e,u_3] & \mathbf{d}[e,u_2] \end{array}$$

In this case, it is worth observing that the second derivative is *not* continuous. The evaluation scheme is given by

$$\begin{array}{ccc} & e & e \\ & \downarrow & \downarrow \\ \mathbf{d}^U[u_0,u_1] & & \\ \mathbf{d}[u_1,u_2] & \mathbf{d}^U[e,u_1] & \\ \mathbf{d}[u_2,u_3] & \mathbf{d}[e,u_2] & \mathbf{d}^U[e,e] \\ \mathbf{d}^V[u_3,u_4] & \mathbf{d}^V[e,u_3] & \mathbf{d}^V[e,e] \end{array}$$

We get two different second derivatives $\mathbf{d}^U[e,e]$ and $\mathbf{d}^V[e,e]$.

The general degree case follows the same pattern.

Now let us consider situations where the common knot s between U and V has multiplicity r with $r \geq 1$. First the quadratic case, now with $r = 2$, i.e., we have a knot sequence $u_0, u_1, u_2, u_2, u_3, u_4$. The first derivative is discontinuous at $s = u_2$ because we have two different values $\mathbf{d}^U[e, u_2]$ and $\mathbf{d}^V[e, u_2]$ instead of a common value $\mathbf{d}[e, u_2]$. So the first derivatives $\mathbf{d}^U[e, u_2]$ and $\mathbf{d}^V[e, u_2]$ exist, they are just different for each blossom. The same is true for the second derivatives. Both cases are illustrated by the following evaluation scheme.

$$\begin{array}{ccc} & e & e \\ & \downarrow & \downarrow \\ \mathbf{d}^U[u_0,u_1] & & \\ \mathbf{d}^U[u_1,u_2] & \mathbf{d}^U[e,u_1] & \\ \mathbf{d}[u_2,u_2] & \mathbf{d}^U[e,u_2] & \mathbf{d}^U[e,e] \\ \mathbf{d}^V[u_2,u_3] & \mathbf{d}^V[e,u_2] & \\ \mathbf{d}^V[u_3,u_4] & \mathbf{d}^V[e,u_3] & \mathbf{d}^V[e,e] \end{array}$$

[2]The symbol e denotes the 1D unit vector $1 - 0$.

Before considering the general case,let us review a cubic example with $n = 3$ and a knot sequence $u_0, u_1, u_2, u_3, u_3, u_4, u_5, u_6$ and $U[u_2, u_3]$, $V = [u_3, u_4]$. Thus $s = u_3$ with multiplicity two. We show the scheme for first, second, and derivatives at s:

$$\begin{array}{cccc}
 & e & e & e \\
 & \downarrow & \downarrow & \downarrow \\
\mathbf{d}^U[u_0, u_1, u_2] & & & \\
\mathbf{d}^U[u_1, u_2, u_3] & \mathbf{d}^U[e, u_1, u_2] & & \\
\mathbf{d}[u_2, u_3, u_3] & \mathbf{d}^U[e, u_2, u_3] & \mathbf{d}^U[e, e, u_2] & \\
\mathbf{d}[u_3, u_3, u_4] & \mathbf{d}[e, u_3, u_3] & \mathbf{d}^U[e, e, u_3] & \mathbf{d}^U[e, e, e] \\
\mathbf{d}^V[u_3, u_4, u_5] & \mathbf{d}^V[e, u_3, u_4] & \mathbf{d}^V[e, e, u_3] & \\
\mathbf{d}^V[u_4, u_5, u_6] & \mathbf{d}^V[e, u_4, u_5] & \mathbf{d}^V[e, e, u_4] & \mathbf{d}^V[e, e, e]
\end{array}$$

Thus there are two different second and third derivatives at u_3 while there is only one first derivative.

One more example: let the knot sequence be $u_0, u_1, u_2, u_3, u_3, u_3, u_4, u_5, u_6$ and $U[u_2, u_3]$, $V = [u_3, u_4]$. Thus $s = u_3$ with multiplicity three. We show the scheme for first, second, and third derivatives at s:

$$\begin{array}{cccc}
 & e & e & e \\
 & \downarrow & \downarrow & \downarrow \\
\mathbf{d}^U[u_0, u_1, u_2] & & & \\
\mathbf{d}^U[u_1, u_2, u_3] & \mathbf{d}^U[e, u_1, u_2] & & \\
\mathbf{d}^U[u_2, u_3, u_3] & \mathbf{d}^U[e, u_2, u_3] & \mathbf{d}^U[e, e, u_2] & \\
\mathbf{d}[u_3, u_3, u_3] & \mathbf{d}^U[e, u_3, u_3] & \mathbf{d}^U[e, e, u_3, & \mathbf{d}^U[e, e, e] \\
\mathbf{d}^V[u_3, u_3, u_4] & \mathbf{d}^V[e, u_3, u_3] & & \\
\mathbf{d}^V[u_3, u_4, u_5] & \mathbf{d}^V[e, u_3, u_4] & \mathbf{d}^V[e, e, u_3] & \\
\mathbf{d}^V[u_4, u_5, u_6] & \mathbf{d}^V[e, u_4, u_5] & \mathbf{d}^V[e, e, u_4] & \mathbf{d}^V[e, e, e]
\end{array}$$

Thus there is no common derivative at u_3. Upon inspection of the blossom values we find that we are dealing with two cubic Bézier curveshaving a common control vertex $\mathbf{d}[u_3, u_3, u_3]$ where they are continuous but (in general) not of higher order continuity.

Now for the general case: let s be of multiplicity r. The highest derivative which agrees for both $\mathbf{d}^U$ and $\mathbf{d}^V$ is $\mathbf{d}[s^{<n>}, e^{<n-r>}]$. Higher order derivatives may only be computed for either $\mathbf{d}^U$ or for $\mathbf{d}^V$.

Conclusion: at a knot of multiplicity r, a B-spline curve is C^{n-r}.