# Convergence of Degree Elevation

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## 1 Basics

Any polynomial may be written as

$$b^n(t) = \sum_{i=0}^{n} b_i B_i^n(t)$$

with

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

being the Bernstein polynomials. It may be written in barycentric notation as

$$b^n(u,v) = \sum_{i+j=n} b_{i,j} B_{i,j}^n(u,v),$$

where $(u,v) = (1-t, t)$ are the barycentric coordinates of the interval $[0,1]$. The Bernstein polynomials now become

$$B_{i,j}^n(u,v) = \frac{n!}{i!j!} u^i v^j; \quad i+j=n.$$

Every degree $n$ polynomial may be written as one of degree $n+1$, having coefficients $b_{i,j}^{n+1}$. They are given by

$$b_{i,j}^{n+1} = \frac{1}{n+1}[i b_{i-1,j} + j b_{i,j-1}].$$

Note that $i+j=n+1$ here.

If a polynomial is degree elevated $r$ times, the new coefficients $b_{\alpha,\beta}^{n+r}$ are given by

$$b_{\alpha,\beta}^{n+r} = \frac{1}{\binom{n+r}{n}} \sum_{i+j=n} \binom{\alpha}{i}\binom{\beta}{j} b_{i,j}.$$

Note that $\alpha+\beta=n+r$.

If we degree elevate a polynomial to degree $d\cdot n$, we have

$$\bar{b}^{dn}_{\alpha,\beta}=\frac{1}{\binom{d(\alpha+\beta)}{n}}\sum_{i+j=n}\binom{d\alpha}{i}\binom{d\beta}{j}b_{i,j}. \tag{1}$$

We use this new notation – employing an overbar – when we degree elevate by multiplying the initial degree by a constant $d$. With this notation, we have $dn\geq\alpha+\beta$.

For any given pair $\alpha,\beta$ of integers, we may degree elevate the given polynomial until a coefficient $b^{n+r}_{\alpha,\beta}$ exists. For example, if $n=3$ and $(alpha,\beta)=(10,20)$, then we degree elevate $r=27$ times until a coefficient $b^{30}_{10,20}$ exists. If we now continue raising the degree to 60, 90, etc., there will be coefficients $\bar{b}^{60}_{10,20},\bar{b}^{90}_{10,20}$ etc.

Convergence of degree elevation is an important theoretical concept; it may be used to prove several properties of curves in Bernstein form, called Bézier curves in Computer Aided Geometric Design.

## 2 Convergence

The coefficients $b_{i,j}$ of a degree $n$ polynomial define a (univariate) piecewise linear function $P^n(u,v)$ with vertices $(i/n,j/n,b_{i,j});i+j=n$, called the control polygon of $b^n$. (Keep in mind that we are using barycentric notation; thus $u+v=1$.) If we degree elevate, we create a sequence of control polygons $P^{n+r}$. These polygons converge to the function that they all define:

$$\lim_{r\to\infty}P^{n+r}(u,v)=b^n(u,v). \tag{2}$$

We now reformulate this convergence claim in order to present a simple proof.

For any given pair of positive integers $\alpha,\beta$, we can assume without loss of generality that a given polynomial has been degree elevated to a degree $k$ such that a coefficient $b^k_{\alpha,\beta}$ exists. Then the convergence statement (2) becomes

$$\lim_{d\to\infty}\bar{b}^{dk}_{\alpha,\beta}=b^k(u,v) \tag{3}$$

with $(u,v)=(\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta})$. An illustration is given in Figure 1.

For our proof of (3), we observe that it is sufficient to prove it for just one generic basis function, $B^k_{\lambda,\mu}$, with $k=\alpha+\beta$, since the function value $b^k(u,v)$ is a finite sum of these basis functions.

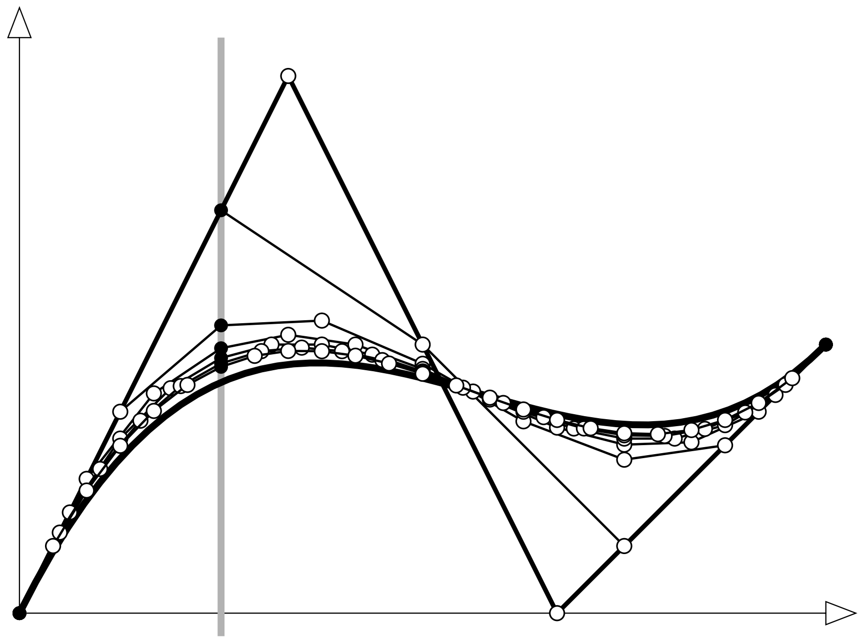

Figure 1: Starting with $n = 3$, we choose $(\alpha, \beta) = (3, 1)$. After elevating to degree 4, the sequence of $\bar{b}^{4d}_{3,1}$ is shown as solid circles, descending towards the curve along the gray vertical line.

If we write out (1) for the case of all $b_{i,j} = 0$ except for some $b_{\lambda,\mu} = 1$, i.e, for a set of coefficients representing the basis function $B^k_{\lambda,\mu}$, we obtain

$$\bar{b}^{dk}_{\alpha,\beta} = \frac{\binom{d\alpha}{\lambda}\binom{d\beta}{\mu}}{\binom{d(\alpha+\beta)}{k}}. \tag{4}$$

This is equivalent to

$$\bar{b}^{dk}_{\alpha,\beta} = \frac{k![d\alpha\cdots(d\alpha-\lambda+1)][d\beta\cdots(d\beta-\mu-k+1)]}{\lambda!\mu![d(\alpha+\beta)\cdots(d(\alpha+\beta)-k+1)]}.$$

Letting $d \to \infty$, we obtain the limit

$$\bar{b}^{\infty}_{\alpha,\beta} = \frac{k!}{\lambda!\mu!}\frac{\alpha^\lambda\beta^\mu}{(\alpha+\beta)^k}$$

or

$$\bar{b}^{\infty}_{\alpha,\beta} = \frac{k!}{\lambda!\mu!}\frac{\alpha^\lambda}{(\alpha+\beta)^k}\frac{b^\mu}{(\alpha+\beta)^k} = B^k_{\lambda,\mu}(u,v)$$

thus proving the convergence claim (3).