Additions, differential geometry

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1 Curvature

Recall that

$$\mathbf{x}' = t'\dot{\mathbf{x}} \quad \text{and} \quad \mathbf{x}'' = (t')^2\ddot{\mathbf{x}} + t''\dot{\mathbf{x}}.$$

Then

$$\begin{aligned}
\kappa(t) &= \|\mathbf{x}''\| \\
&= \|\mathbf{x}' \wedge \mathbf{x}''\| \\
&= \|\frac{\dot{\mathbf{x}}}{\dot{s}} \wedge (\frac{\ddot{\mathbf{x}}}{\dot{s}^2} + t''\dot{\mathbf{x}})\| \\
&= \frac{\|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}\|}{\dot{s}^3} \\
&= \frac{\|\dot{\mathbf{x}} \wedge \ddot{\mathbf{x}}\|}{\|\dot{\mathbf{x}}\|^3}
\end{aligned}$$

To see the step from the third to the fourth equation, recall that

$$\|\mathbf{a} \wedge (\mathbf{b} + \alpha\mathbf{a})\| = \|\mathbf{a} \wedge \mathbf{b}\|.$$

2 Frenet-Serret equations

Since $\mathbf{m}\mathbf{b} = 0$, it follows that $\mathbf{m}'\mathbf{b} = -\mathbf{m}\mathbf{b}'$, and analogous expressions for the remaining pairs of vectors.

Every vector $\mathbf{v}$ may be written in terms of the orthonormal system of vectors $\mathbf{t}, \mathbf{m}, \mathbf{b}$:

$$\mathbf{v} = (\mathbf{v}\mathbf{t})\mathbf{t} + (\mathbf{v}\mathbf{m})\mathbf{m} + (\mathbf{v}\mathbf{b})\mathbf{b}.$$

If we apply this to the three vectors $\mathbf{t}', \mathbf{m}', \mathbf{b}'$, then we get

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{m}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{t}'\mathbf{m} & \mathbf{t}'\mathbf{b} \\ \mathbf{m}'\mathbf{t} & 0 & \mathbf{m}'\mathbf{b} \\ \mathbf{b}'\mathbf{t} & \mathbf{b}'\mathbf{m} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{m} \\ \mathbf{b} \end{bmatrix}.$$

The zero entries stem from the fact that $\mathbf{t}'\mathbf{t} = 0$ etc. Since $\mathbf{t}' = \kappa\mathbf{m}$, the matrix elements $(1,3)$ and $(3,1)$ also have to vanish. Thus if we define

$$\tau = \mathbf{m}'\mathbf{b},$$

the above matrix equation reduces to

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{m}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{m} \\ \mathbf{b} \end{bmatrix}$$

These are the Frenet-Serret equations.