Circles and polynomials

Gerald Farin

November 17, 1999

Why can a part of a circle not be represented by a (nonrational) polynomial curve segment?
First, a simple observation. If

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots a_0,$$

then $f^2(t)$ has a leading term of $a_n^2 t^{2n}$, i.e., a leading positive coefficient.
Now assume that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{1}$$

traces out a circle and that $x(t)$ and $y(t)$ are polynomials of respective degrees n_x and n_y. Clearly we must have

$$x^2(t) + y^2(t) \equiv 1.$$

Without loss of generality, let $n_x \geq n_y$. Then the leading term of the polynomial $x^2(t) + y^2(t)$ is of the form αt^{2n_x}; $\alpha > 0$. For this polynomial to equal 1, the only conclusion is $n_x = 0$. But a degree 0 polynomial curve canot represent a circle, a contradiction to the assumption that (1) could be a polynomial representation of a circle.