Curvature combs and curvature plots

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1. Introduction

Many objects in Industrial or Conceptual Design rely on an initial set of “feature curves”. These are the essence of the shape information for the object’s appearance.

For many applications, these curves are planar, for example flow lines of a ship hull or a grid of cross sections of the roof of a car. For others, they will be 3D space curves, such as the outline of a car’s hood.

Either type of curve has to be “fair” or “pleasant”, thus giving rise to acceptable surfaces for the object to be designed. The question arises as to what constitutes a fair curve. Most CAD systems seem to rely on some sort of curvature information for this, and the prevailing tools are curvature combs and curvature plots. A comparison of these tools is the topic of the present paper.

2. Definitions

Let $\mathbf{x}(t)$ be a 3D parametric curve. We will use the following quantities from differential geometry (see [1]):

- the first two parametric derivatives $\dot{\mathbf{x}}(t)$ and $\ddot{\mathbf{x}}(t)$,
- the binormal vector (with $\| \cdot \|$ denoting normalizing to unit length)
  $$\mathbf{b}(t) = \frac{\dot{\mathbf{x}}(t) \wedge \ddot{\mathbf{x}}(t)}{\| \cdot \|},$$
- and the normal vector
  $$\mathbf{n}(t) = \frac{\mathbf{b}(t) \wedge \dot{\mathbf{x}}(t)}{\| \cdot \|}.$$  

3. Curvature plots

The curvature of a 3D curve is given by

$$\kappa(t) = \frac{\| \dot{\mathbf{x}} \wedge \ddot{\mathbf{x}} \|}{\| \mathbf{x} \|^3}.$$  

For the special case of planar curves, we may define signed curvature

$$\kappa_s(t) = \frac{\operatorname{det}[\mathbf{x}, \dot{\mathbf{x}}]}{\| \dot{\mathbf{x}} \|^3},$$

where $\operatorname{det}$ is the determinant. Note that 3D curvature is always nonnegative whereas the 2D case allows for mixed-sign curvatures. Thus for 2D curves, the notion of “inflection point” (sign change of $\kappa_s$) makes sense, but it does not for 3D curves.

The graph of $\kappa(t)$ or $\kappa_s(t)$ is referred to as the curvature plot. What makes a “good” curvature plot? It is generally agreed that good plots have only very few monotone segments, corresponding to low frequency behavior of the curvature. High frequency curvature behavior indicates “unpleasant” curves. Also, a good curvature plot lacks “corners” (tangent discontinuities). See [2,3].

4. Combs

The curvature comb of a curve is defined as:

$$\mathbf{c}(t) = \mathbf{x}(t) - d \cdot \kappa(t) \mathbf{n}(t).$$  

(1)

If we discretize $\mathbf{c}$, then we get the appearance of a comb distributed along $\mathbf{x}$, as illustrated in Fig. 1, hence the term “curvature comb”. In our examples, comb curves are shown in red; the input curves are in black. The scaling factor $d$ is needed to bring out the salient features of the comb.
Wemightusearclength$s$insteadoftheparameter$t$.Then,thecurvaturecombchangesappearancebecauseoftheresulting changeofsamplingpointsonthecurve.Usingarclengthgives amorefine-tunedtool,buttithardertocompute.Plotting combteethwitharclengthspacingtypicallygivesamoreeven distributionalongthecurvesbyeliminatingeffectsofthecurve traversalaccordingtotheparameter$t$.Ifthistransformationisonly $C^0$,thecombwouldnotchange,butthecurvatureplotwill.

Forthecombplotstofollow,werendercombsasparametric surfaces
\[ c(t,u) = x(t) - d \cdot u \cdot \kappa(t) \mathbf{n}(t) \] (2)
with$t$rangingoverthecurve’sdomainand$0 \leq u \leq 1$.This eliminatesconsiderationofhowtoplotafamilyofcombteeth.

Combsareencounteredas “evolutes”indifferentialgeometry, see [4–6].

In differential geometry, the Frenet–Serret equations are given by
\[ \mathbf{t}' = -\kappa \mathbf{n} \]
\[ \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b} \]
\[ \mathbf{b}' = -\tau \mathbf{n} \]
where the prime denotes differentiation with respect to arc length and$\tau$is the torsion. See [4,1].

Weseethat$\mathbf{n}'$depends on both curvature and torsion, hence does$\mathbf{n}$, and the name “curvaturecomb” really should be “curvature–torsioncomb”for space curves. Here, we simplyuse “comb”.

Thecombinationofcurvatureandtorsioninformationofthe combmaygiveda designer amore complex tool than just usingthecurvatureplot. We point out here that for true 3D curves, this may be more information than is useful in a typical design situation.

5. Usage

Many commercial CAD systems (such as Alias, Catia, or NX) employcombsfor curve shape analysis. A more elementary approach is to utilize the curvature plot. We compare both in Figs. 2and 3.

The comb plot is the result of superimposing$\mathbf{c}$over$\mathbf{x}$. Its appearance is thus a combination of two curve shapes, making it harder to judge than the simple curvature plot. While space curves cannot have inflection points, in some views combs may suggest nonexistent inflection points. Thus any interpretation of the 3D comb’s shape is view-dependent, hardly making for an objective analysis tool. In addition, the constant$d$in (1) influences the comb’s appearance—a judicial choice is needed for every case. In fact, different parts of the curve might necessitate different values of$d$.

In many applications, only the shape of planar curves is analyzed. In many others, consideration of 3D curves is necessary. For example, the feature curves on the hood of a car are 3D, and a
2D analysis is inappropriate. If we were to design a roller coaster, it is also obvious that planar curves do not enter the picture.

The comb plot allows for a natural correspondence between the comb and the curve. In a curvature plot, points on the curvature graph are disconnected from points on the curve. However, all points with $κ'(t) = 0$ can be marked on the curve.

Curvature plots lend themselves to the display of curvature or radius of curvature numerical values. Curvature plots may require additional graphics tools such as extra window environments, which in all fairness will add complexity to the user interface.

One frequently encounters the problem of considering several curves at once. These are typically interrelated (parallel) surface 2D cross sections. The analysis of this whole family of curves is important, and just a set of curvature plots does not give adequate information. More research is needed here.

6. Conclusion

We explored some properties of combs. For 3D curves, they are themselves 3D objects offering a complex mix of information and require a 3D visualization environment. By comparison, curvature plots are simpler and, as a consequence, omit 3D information that might be misleading to the designer.

Based on our studies, we recommend using curvature plots for true space curves and combs for nearly planar curves. Of course, the actual choice should be application-dependent. Curvature plots are best for the initial design of one curve. Combs excel when multiple curves are displayed.

References