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Rational quadratic circles are parametrized by chord length

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Abstract

We show that the chord length parameter assignment is exact for circle segments in standard rational quadratic form.

Keywords: Chord length parametrization; Rational quadratics; Standard form

1. Rational quadratic circles

An arc of a circle may be written as a rational quadratic:

$$\mathbf{p}(t) = \frac{\mathbf{c}_0 B_0^2(t) + v_1 \mathbf{c}_1 B_1^2(t) + \mathbf{c}_2 B_2^2(t)}{B_0^2(t) + v_1 B_1^2(t) + B_2^2(t)} \tag{1}$$

where the control points $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ form an isosceles triangle with base $\mathbf{c}_0, \mathbf{c}_2$, see (Farin, 2001) or (Farin, 1999). The $B_i^2(t)$ are the quadratic Bernstein polynomials. If the base angle of the triangle is α, then the weight of $\mathbf{c}_1$ is $v_1 = \cos\alpha$. The control points $\mathbf{c}_0$ and $\mathbf{c}_2$ have unit weights—this is referred to as the *standard form* of a rational quadratic curve. The three points $\mathbf{c}_0, \mathbf{p}(t), \mathbf{c}_2$ correspond to parameter values $0, t, 1$, respectively.

For more details, see (Farin, 2001). For an illustration, see Fig. 1.

While the rational quadratic (1) does describe a circular arc, its parametrization is not the arc length one (for a proof, see (Farouki and Sakkalis, 1991)). This is sometimes noted as a drawback. We now show that the parametrization has a different property instead: it is the chord length parametrization.

2. Chord length

An important concept in the theory of interpolating curves is that of *chord length*, see (Farin, 2001).[1] Borrowing from that notion, we define: a point $\mathbf{p}(t)$ on a parametric curve $\mathbf{p}(t)$; $a \leqslant t \leqslant b$ has (continuous) chord length $\mathrm{chord}(t)$

$$\mathrm{chord}(t) = \frac{\|\mathbf{p}(t) - \mathbf{p}(a)\|}{\|\mathbf{p}(t) - \mathbf{p}(a)\| + \|\mathbf{p}(t) - \mathbf{p}(b)\|}. \tag{2}$$

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[1] An interpolating curve which has chord length parameters assigned to the data points is often referred to as being “chord length parametrized”. This is not true using definition (2).

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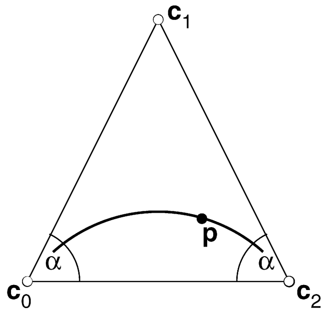

Fig. 1. A rational quadratic circular arc.

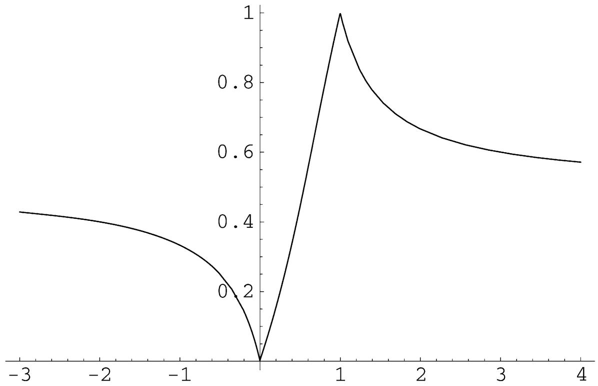

Fig. 2. Plot of chord(t) vs. t.

Now suppose we are given three points $\mathbf{c}_0, \mathbf{p}(t), \mathbf{c}_2$ with $\mathbf{p}(t)$ a point on the rational quadratic circle (1). The chord length parameter chord(t) for $\mathbf{p}(t)$ is given by

$$\text{chord}(t) = \frac{\|\mathbf{p}(t) - \mathbf{c}_0\|}{\|\mathbf{p}(t) - \mathbf{c}_0\| + \|\mathbf{p}(t) - \mathbf{c}_2\|}. \tag{3}$$

The result of this short note is

$$\text{chord}(t) = t \quad \text{for } t \in [0, 1], \tag{4}$$

i.e., rational quadratic circle segments in standard form are (parametrized by) chord length.

Proof. See Mathematica code in Appendix A. A different proof, based on geometric and trigonometric arguments, is given in (Sabin and Dodgson, 2005).

It follows that the rational quadratic standard parametrization of a circular arc is the only one which is parametrized by chord length.

Fig. 2 shows chord(t) plotted as a function of t. Note that the identity (4) only holds for $t \in [0, 1]$.

The remaining part of the full circle (the complementary segment) is obtained by replacing v_1 by $-v_1$. Eq. (4) also holds for this complementary segment.

There are many different parametrizations of a circular arc, most notably

$$\mathbf{x}(s) = \begin{bmatrix} \cos(s) \\ \sin(s) \end{bmatrix}$$

in which the parameter s is the arc length parameter. It is straightforward to show (by way of counterexample) that his parametrization is not chord length. Other parametrizations of circular arcs do exist: (Chou, 1995), (Fiorot et al., 1997), or (Bangert and Prautzsch, 1997), none of these has the cord length property either.

Appendix A. Proof by Mathematica

We define the recursive de Casteljau algorithm and its rational version `decas` and `rdecas`:

```
decas[b_, t_, 0, i_] := b[[i + 1]];
decas[b_, t_, r_, i_] := (1 - t)
decas[b, t, r - 1, i] +
    t decas[b, t, r - 1, i + 1];

rdecas[b_, w_, t_, 0, i_] := b[[i + 1]];
 rdecas[b_, w_, t_, r_,
   i_] := ((1 - t) decas[w, t, r - 1, i]
    rdecas[b, w, t, r - 1, i] +
    t decas[w, t, r - 1, i + 1] rdecas[b, w, t, r - 1, i + 1])/
decas[w, t, r, i];
```

Without loss of generality, we restrict our control polygon to be $[-1, 0]$, $[0, h]$, $[1, 0]$ with a variable h:

```
 a[h_] :={{-1,0},{0, h}, {1, 0}};
 (*triangle only depends on parameter h*)
v[h_] := {1, {1, h}.{1, 0}/Norm[{1, h}],1}; (* weights *)
```

We define the chord length parametrization:

```
clength[h_, t_] :=
Norm[rdecas[a[h], v[h], t, 2, 0] - {-1, 0}]/(Norm[
        rdecas[a[h], v[h], t, 2, 0] - {-1, 0}] +
      Norm[rdecas[a[h], v[h], t, 2, 0] - {1, 0}]);
```

The proof of our statement is now to simplify the (rather complex) expression for chord(t) to t:

```
Simplify[clength[h, t], 0 < t < 1]

t
```

Changing the sign of the second entry of $v[h]$, i.e., considering the complementary segment, yields the same `Simplify` result.

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