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# Dimensions of spline spaces over unconstricted triangulations

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**Abstract**

One of the puzzlingly hard problems in Computer Aided Geometric Design and Approximation Theory is that of finding the dimension of the spline space of $C^r$ piecewise degree $n$ polynomials over a 2D triangulation $\Omega$. We denote such spaces by $\mathcal{S}_n^r(\Omega)$. In this note, we restrict $\Omega$ to have a special structure, namely to be *unconstricted*. This will allow for several exact dimension formulas.


*Keywords:* Bivariate spline spaces; Dimensions; Minimal determining sets; Bernstein–Bézier form; Unconstricted triangulations

## 1. Introduction and definitions

Splines, i.e., piecewise polynomials, form linear spaces that have a very simple structure in the univariate case. In particular, it is trivial to determine the dimensions of these spaces. The bivariate case is much harder. Here, the polynomial pieces are defined over triangles which form a triangulation of a subset of 2-space. The dimension of these spaces depends not only on the number of triangles, the degree and smoothness of the splines, but also on the geometry of the triangulation. For the general case, no dimension formula is known; there are results which constrain degree $n$ and smoothness $r$, see [1,2,4,10,11]. In this paper, we constrain the type of triangulation, but are able to consider unconstrained degree and smoothness.

We now give some basic definitions.

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A 2D *triangulation* $\Omega$ of a 2D point set $\{\mathbf{p}_i\}$ is a collection of triangles $T_j$ such that

(a) the triangle vertices consist of the $\mathbf{p}_i$,
(b) the interiors of any two triangles do not intersect,
(c) if two triangles are not disjoint, then they share either an edge or a vertex.

A *subtriangulation* $\Omega'$ of a triangulation $\Omega$ is a triangulation satisfying

$$T \in \Omega' \Rightarrow T \in \Omega.$$

Two triangles are called *neighbors* if they share a common edge.

The *boundary* of a triangulation is the set of all triangle (boundary) edges which are not shared by two triangles. The corresponding triangles are called *boundary triangles*. The boundary edge vertices are called *boundary vertices*. All other triangle vertices are called *interior vertices*.

Let $\boldsymbol{p}$ be a vertex of a triangle in a triangulation $\Omega$. Let $k$ be the number of all edges emanating from $\boldsymbol{p}$. Then $k$ is called the *valence* of $\boldsymbol{p}$.

The set of all $C^r$ piecewise bivariate polynomials of degree $n$ over $\Omega$ is defined as

$$\mathscr{S}_n^r(\Omega) = \{s \in C^r(\Omega) : s|_{T_i} \in \mathbb{P}^n\},$$

where $\mathbb{P}^n$ denotes the set of all bivariate polynomials of degree $n$.

We assume the reader is familiar with the theory of Bernstein–Bézier triangles, see [5] or [6]. For the sake of completeness, a Bézier triangle of degree $n$ is given by

$$b(u, v, w) = \sum i + j + k = n \frac{n!}{i!j!k!} u^i v^j w^k b_{ijk},$$

where $(u, v, w)$ are the barycentric coordinates of a point in a triangle, and the $b_{ijk}$ are the *Bézier ordinates* of the bivariate polynomial $b$.

We will need the notion of *minimal determining sets*.[1] Let $M$ be a subset of all Bézier ordinates in $\mathscr{S}_n^r(\Omega)$ and assign function value 0 to all elements of $M$. The $C^r$ conditions relate these to the remaining Bézier ordinates. If the conditions force all remaining Bézier ordinates to be zero, then $M$ is called a determining set. If $M$ has the smallest possible number of Bézier ordinates, then it is a minimal determining set. Clearly, the size of $M$ equals dim $\mathscr{S}_n^r(\Omega)$.

## 2. Constricted triangulations

A *constricted triangulation* is a 2D triangulation containing a subtriangulation with all boundary vertices having valence 4 or more.

The motivation for this term is as follows. The average valence of an interior vertex in a triangulation is 6. The average valence of a boundary vertex is between 3 and 4. Thus a (sub)triangulation where *every* boundary vertex has a valence 4 or more has above average high valence boundary vertices, “constricting” the (sub)triangulation.

[1] These were introduced by Alfeld and Schumaker [1]. See also Peter Alfeld’s web site http://www.math.utah.edu/alfeld.

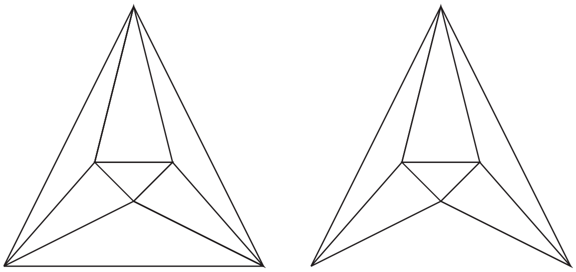

Fig. 1. The (constricted) Morgan–Scott triangulation, left. Right: an unconstricted triangulation.

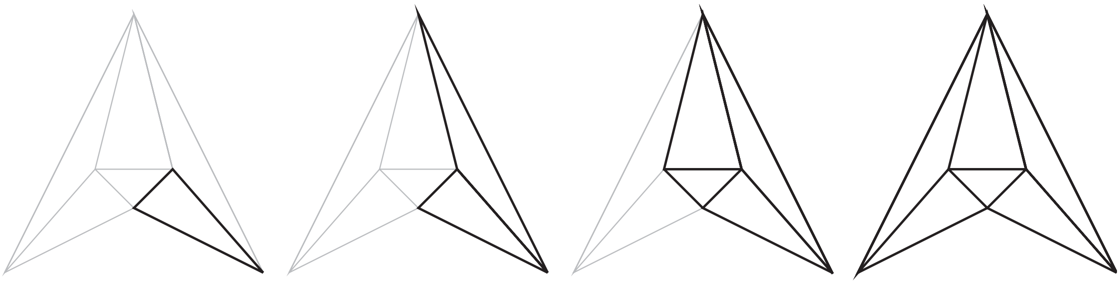

Fig. 2. The flap-and-pair construction. From left to right: start with one triangle, add flap, add pair, add pair.

In this paper, we only consider triangulations with nonsingular vertices, i.e., triangulations without vertices with collinear edges emanating from them. (That more general case is not expected to significantly add to the results derived here.)

The simplest example of a constricted triangulation is the so-called Morgan/Scott triangulation, see Fig. 1. Any triangulation may be transformed into a constricted one by splitting every boundary triangle into three triangles by a centroid split.

An unconstricted triangulation is one which is not constricted, i.e., one not containing a constricted subtriangulation. An example is shown in Fig. 1.

It may be constructed in the following flap-and-pair manner.

Start: one triangle.

Recursive step: assuming a subtriangulation has already been constructed, we may extend the triangulation using two operations on boundary vertices.

(1) Form a new boundary point by adding a flap (forming a new triangle from one boundary edge and a point outside the current triangulation). This operation adds one boundary vertex and leaves the number of interior vertices unchanged.
(2) Form a new interior point by adding a pair of triangles. This adds one interior vertex and leaves the number of boundary vertices unchanged.

See Fig. 2 for an example of the flap-and-pair process.

While this constitutes a way for constructing an unconstricted triangulation, there is also an easy way to check if a given triangulation is unconstricted: For every boundary vertex with valence two, remove the corresponding triangle (a flap). For every boundary vertex of valence three, remove the corresponding pair of triangles. Continue as long as possible. If this procedure ends with only one triangle, we had an unconstricted triangulation.

## 3. The $C^1$ cubic case

We now show the following:

**Lemma.** *For an unconstricted triangulation $\Omega$, the dimension of all $C^1$ piecewise cubics over it is given by*

$$\dim\ \mathscr{S}_3^1(\Omega) = 3b + 2i + 1, \tag{1}$$

*where i is the number of interior vertices and b is the number of boundary vertices.*

**Proof.** This result follows from a more general one given below; however it is of interest in its own right.

Clearly it holds for a triangulation $\Omega$ with only one triangle: then $\dim\ \mathscr{S}_3^1(\Omega) = 10$. For an inductive proof, assume (1) holds for a subtriangulation. Adding a flap is consistent with (1): it adds three degrees of freedom. Adding a pair of triangles is consistent with it as well: it adds two degrees of freedom. Note that this proof is constructive in that it produces a minimal determining set for $\mathscr{S}_3^1(\Omega)$. Such a set is shown in Fig. 3 where a minimal determining set is noted by solid black circles. □

Eq. (1) has been conjectured in [10] (for general triangulations) as early as 1973; see also [9]. A more recent treatment is given in [3, pp. 401 and 404]. Here we were able to show that the conjecture does hold for unconstricted triangulations. In the next section, we will generalize this approach.

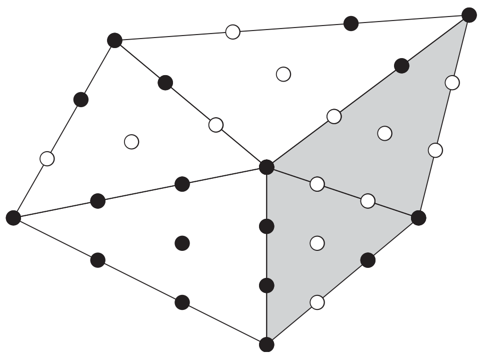

Fig. 3. Constructing a spline space.

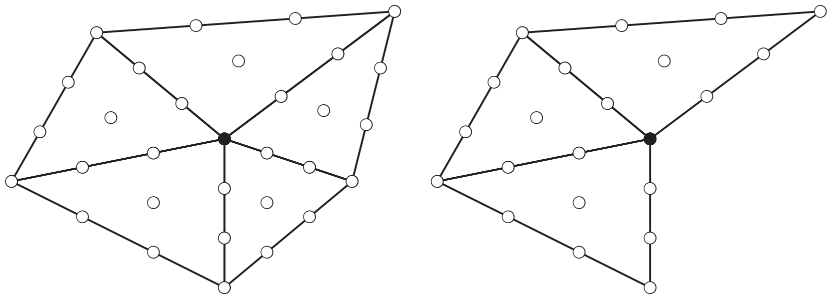

Fig. 4. The two triangulations $\mathbf{v}^\star$ (left) and $\mathbf{v}^{\star-2}$ (right). Bézier ordinates for the cubic case are shown.

## 4. Spline spaces over stars

A *star* $\mathbf{v}^\star$ is a triangulation consisting of one interior vertex $\mathbf{v}$ with all boundary vertices having valence 3. Let us assume $\mathbf{v}^\star$ has $b$ boundary vertices, i.e., $\mathbf{v}$ has valence $b$. Assume that no two interior edges are collinear. Let $\mathscr{S}_n^r(\mathbf{v}^\star)$ be the space of all $C^r$ piecewise polynomials defined over $\mathbf{v}^\star$. Then, with the notation $\overline{x} = (x+1)(x+2)/2$,[2] Schumaker [8] proves that

$$\dim\ \mathscr{S}_n^r(\mathbf{v}^\star) = (\overline{n-1-r})b + \overline{r} + \sum_{i=1}^{n-r} (r+i+1-ib)_+. \tag{2}$$

Now consider a triangulation $\mathbf{v}^{\star-2}$ which is obtained from $\mathbf{v}^\star$ by removing two adjacent triangles (i.e., a pair of triangles). The triangulation $\mathbf{v}^\star$ has no interior point and $b$ boundary vertices. For an illustration of the two triangulations and Bézier ordinates for the cubic case, see Fig. 4.

We now have the following:

**Lemma.**

$$\dim\ \mathscr{S}_n^r(\mathbf{v}^{\star-2}) = \overline{n} + (b-3)\overline{n-r-1}. \tag{3}$$

**Proof.** We prove (3) by recursively finding a minimal determining set. If $\mathbf{v}^{\star-2}$ consists of only one triangle, then $b=3$ and the lemma holds. Building $\mathbf{v}^{\star-2}$ by successively adding flaps, we add $\overline{n-r-1}$ Bézier ordinates for each added flap. Since there are $b-3$ flaps to be added, (3) is proved. $\square$

The proof is constructive in that it produces a minimal determining set for $\mathscr{S}_n^r(\mathbf{v}^{\star-2})$.

The difference $\delta_n^r(b)$ between these two dimensions only depends on the numbers $b, n, r$:

$$\begin{aligned}\delta_n^r(b) :=&\ \dim\ \mathscr{S}_n^r(\mathbf{v}^\star) - \dim\ \mathscr{S}_n^r(\mathbf{v}^{\star-2}) \\ =&\ \overline{r} + \sum_{k=1}^{n-r} (r+i+1-kb)_+ - \overline{n} + 3(\overline{n-r-1}).\end{aligned} \tag{4}$$

Note that $\delta_n^r(b) = \delta_n^r(\mathrm{val}(\mathbf{v}))$ with $\mathrm{val}(\mathbf{v})$ denoting the valence of $\mathbf{v}$ in $\mathbf{v}^\star$.

[2] Recall that $\overline{n}$ is the dimension of all bivariate polynomials of degree $n$.

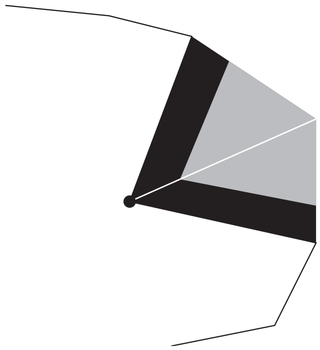

Fig. 5. Adding a pair of triangles. Black: the $r+1$ rows of Bézier ordinates which are determined by $C^r$ conditions. Gray: the remaining $(n-r)^2$ Bézier ordinates.

The triangulation $\mathbf{v}^{\star}$ is obtained from $\mathbf{v}^{\star-2}$ by adding a pair of triangles. This pair of triangles has $r+1$ rows of Bézier ordinates determined by the $C^r$ conditions connecting them to the Bézier ordinates of $\mathscr{S}_n^r(\mathbf{v}^{\star-2})$, shown in black in Fig. 5. This leaves $(n-r)^2$ Bézier ordinates undetermined, shown in gray in the same figure. For these, $(n-r)^2-\delta_n^r$ are determined by the $C^r$ conditions, and they may be chosen arbitrarily if $\delta_n^r \geqslant 0$. These Bézier ordinates are then added to the minimal determining set for $\mathscr{S}_n^r(\mathbf{v}^{\star-2})$, thus forming a minimal determining set for $\mathscr{S}_n^r(\mathbf{v}^{\star})$. Fig. 3 shows a minimal determining set marked by black points.

If $\delta_n^r < 0$, we cannot add any elements to the minimal determining set of $\mathscr{S}_n^r(\mathbf{v}^{\star-2})$. In addition, the minimal determining set for $\mathscr{S}_n^r(\mathbf{v}^{\star-2})$ cannot be used for finding a minimal determining set for $\mathscr{S}_n^r(\mathbf{v}^{\star})$.

## 5. Spline spaces over unconstricted triangulations

Now let $\Omega$ be an unconstricted triangulation and let $\mathbf{I}$ be the set of its interior vertices, consisting of $i$ interior vertices. From the flap-and-pair construction of an unconstricted triangulation, we arrive at the following dimension formula.

**Theorem.** *If $\delta_n^r(\mathrm{val}(\mathbf{v})) \geqslant 0$ for all $\mathbf{v} \in \mathbf{I}$, and if $\Omega$ is an unconstricted triangulation, then*

$$\dim \mathscr{S}_n^r(\Omega) = \overline{n} + b(\overline{n-r-1}) + \sum_{\mathbf{v}\in\mathbf{I}} \delta_n^r(\mathrm{val}(\mathbf{v})). \tag{5}$$

**Proof.** We recursively construct a minimal determining set for $\mathscr{S}_n^r(\Omega)$. Eq. (5) holds for $i=0$ and $i=1$ interior points and for any number of boundary points $b$.

We now assume the theorem holds for triangulations with $i$ interior vertices. We may create a new triangulation by adding a flap. The dimension of the corresponding space grows in accordance to (5).

We may increase $i$ to $i+1$ by adding to triangles as in the construction of Section 4. Let the newly formed interior point have valence $b$. Since we assume $\delta_n^r(b) \geqslant 0$, we may add $\delta_n^r$ Bézier ordinates to the minimal determining set by selecting $\delta_n^r$ Bézier ordinates among the $(n-r)^2$ unknown Bézier ordinates. $\square$

**Corollary.** *If* $\delta_n^r(\mathrm{val}(\mathbf{v})) < 0$ *for at least one* $\mathbf{v} \in \mathbf{I}$, *and if* $\Omega$ *is an unconstricted triangulation*, *then*

$$\dim\ \mathscr{S}_n^r(\Omega) \geqslant \overline{n} + b(\overline{n-r-1}) + \sum_{\mathbf{v}\in\mathbf{I}} \delta_n^r(\mathrm{val}(\mathbf{v})). \tag{6}$$

**Proof.** As we recursively construct $\Omega$, we encounter at least one case where the above construction fails, namely the case where $\delta_n^r < 0$. In this case, we cannot add any elements to the minimal determining set. This does not necessarily imply that the dimension drops—there may be linearly independent $C^r$ conditions among all conditions governing $S_n^r(\Omega)$. Hence we can only produce a lower bound for dim $\mathscr{S}_n^r(\Omega)$. □

## 6. Special cases

We now discuss some special cases.

For $n = 5$, $r = 3$, we obtain (setting $\beta = \mathrm{val}(\mathbf{v})$).

$$\delta_5^3(\beta) = (5-\beta)_+ - 2 \geqslant 0 \Rightarrow \beta = 3.$$

For this to be nonnegative, we need $\beta = 3$ for all $\mathbf{v} \in \mathbf{I}$. There is only one such triangulation, namely one triangle split into three triangles at an interior point. This is known as the Clough–Tocher split.

For $n = 6$, $r = 4$, we have

$$\delta_6^4(\beta) = (6-\beta)_+(7-2\beta)_+ - 4.$$

This is nonnegative for no value of $\beta$; hence we can only give a lower bound for dim $\mathscr{S}_6^4(\Omega)$ for unconstricted triangulations $\Omega$.

For the case $r = n-1$, we see that $\delta_n^{n-1}(\beta) \geqslant 0$ reduces to $\beta = 3$. Again, the Clough–Tocher split is the only triangulation admitting this configuration.

For the case $r = 1$, we have

$$\delta_n^1(\beta) = 2 + \sum_{k=1}^{n-1} (2 + k - k\beta)_+ + n^2 - 3n.$$

Since $\beta \geqslant 3$, this reduces to $\delta_n^1(\beta) = 2 - 3n + n^2$ which is independent of $\beta$ and nonnegative for all $n$. Hence for $r = 1$, the dimension given by (5) holds for all unconstricted triangulations.

Finally, the case $n = 2r$. Then

$$\delta_{2r}^r(\beta) = \sum_{k=1}^{r} (r + k + 1 - k\beta)_+.$$

This is nonnegative for all $r$ and thus (5) does give the correct dimension. For arbitrary triangulations, only the bound $n > 3r + 2$ is known to allow for a dimension formula, see [7].

## 7. Conclusions

We were able to give exact dimensions for some spline spaces defined over unconstricted triangulations. It is hoped that this partial result will be helpful in finding a general dimension formula.

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