Geometric Hermite Interpolation with Circular Precision

Gerald Farin
Computer Science
Arizona State University

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Abstract

We present several Hermite-type interpolation methods for rational cubics. In case the input data come from a circular arc, the rational cubic will reproduce it.

keywords: Hermite interpolation, rational cubics, circular precision.

1 Introduction

We define the *geometric Hermite* design problem as: given two points $\mathbf{b}_0$, $\mathbf{b}_3$ and two unit tangent directions $\mathbf{v}_0, \mathbf{v}_1$ at those points, find a curve that meets those constraints. Traditionally, a parametric polynomial cubic is used to solve this problem. In Bézier form, it is defined by four control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, see Fig. 1. Thus one needs to find "good" locations for $\mathbf{b}_1, \mathbf{b}_2$. There are many more or less ad-hoc approaches in the literature (not all using the Bézier form), see [1], [12], [2]. None of these methods are capable of achieving geometric objectives; rather, they are based on approximation-theoretic motivations.

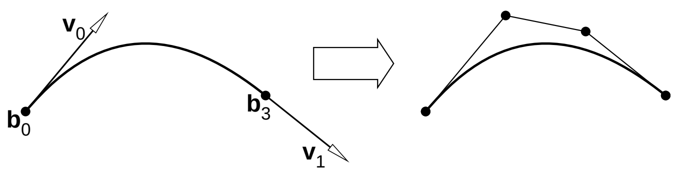

Figure 1: The geometric Hermite interpolation problem. Left: the given input data. Right: a solution in Bézier form.

The standard cubic Hermite interpolant prescribes derivative vectors $\mathbf{h}'(0)$ and $\mathbf{h}'(1)$ at $\mathbf{b}_0$ and $\mathbf{b}_1$ and then finds

$$\mathbf{b}_1 = \mathbf{b}_0 + \frac{1}{3}\mathbf{h}'(0), \quad \mathbf{b}_2 = \mathbf{b}_3 - \frac{1}{3}\mathbf{h}'(1).$$

In a typical design situation, one is not given exact derivative vectors but rather the unit tangent directions $\mathbf{v}_0$ and $\mathbf{v}_1$. This is illustrated in the font design example of Fig. 2. How to find the Bézier points and weights in the right of that figure is the subject of this paper.

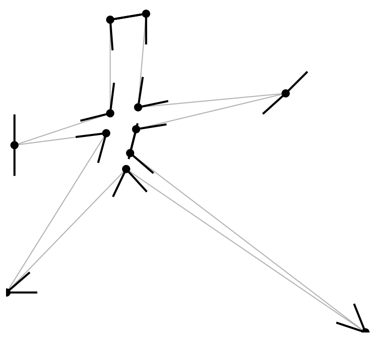

Figure 2: Font design. Left: the given input data. Right: a solution in Bézier form.

In this paper, we introduce rational cubics for solving the geometric Hermite problem. Utilizing the concept of *circular precision*, we arrive at a class of rational cubic curves which reproduce circular arcs where possible.

Specifically, we address the following problems:

- interpolate to two points and tangent vectors (Section 5).
- interpolate to two points and unit tangent vectors (Section 6)

2 History

The use of rational curves (in Bézier form) in CAGD goes back to R. Forrest [9]. This work was inspired by his mentor S. Coons who had developed an interest in rational techniques early on, see [4]. These early developments focused on exhibiting properties of rational curves and surfaces, and paid less attention to actual implementation issues. A similar comment applies to another work inspired by S. Coons, namely K. Vesprille's thesis introducing rational B-splines [14].

The main use of rational schemes, starting from the mid-80s, was their ability to represent both polynomial (or piecewise polynomial) entities as well as conics. This was when, initiated by Boeing and SDRC, the term NURBS (non-uniform rational B-splines) was coined.

In addition to offering a universal data structure, rational schemes offer the additional flexibility of *weights* (see below) over nonrational schemes where those weights are forced to unity. Since the weights enter the definition of a rational curve in a highly nonlinear fashion, attempts to use them for shape optimization purposes have been scarce; we note [3], [11] as early efforts. Both articles treat the weights as unknowns and fix them by minimizing an energy functional aiming at curvature optimization.

A generalized Hermite problem is defined by requiring interpolation to curvatures at curve end points. This yields algorithms for finding weights of rational cubics, see [5] and [10].

3 Shape Measure

When introducing a new curve scheme, an argument for its relevance and novelty needs to be made. We offer the following. Curvature is the universal shape measure for curves, often encountered as the "bending energy" e:

$$e(\mathbf{x}) = \int_0^1 [\kappa(\tau)]^2 \mathrm{d}\tau$$

where $\mathbf{x}$ is a parametric curve and κ is its curvature.

For "pleasant" curve shapes, a different quantity is more important. It is given by

$$s(\mathbf{x}) = \int_0^1 [\kappa'(\tau)]^2 \mathrm{d}\tau, \tag{1}$$

meaning that the change in curvature is more important than the magnitude of curvature. According to P. Bézier,[1] even a discontinuous jump in curvature κ is acceptable as long as the slope κ' is continuous.

The quantity $s(\mathbf{x})$ is zero for circular arcs when τ is the arc length parameter.[2] Thus striving for circular arcs appears desirable wherever possible. In this paper, we show how to construct rational cubics which are close to circles, thus aiming to minimize (1) with a geometric rather than a variational approach.

[1] Private conversation, 1991.

[2] It has been shown in [8] that a rational curve's parameter cannot be the *exact* arc length parameter.

4 Rational Cubic Circles

In this section, we investigate the interplay between rational cubic and quadratic representations of a circular arc – this will later facilitate our work with rational cubics which are close to circles.

A 2D rational cubic Bézier curve is given by

$$\mathbf{x}(t) = \frac{\mathbf{b}_0 B_0^3(t) + w_1 \mathbf{b}_1 B_1^3(t) + w_2 \mathbf{b}_2 B_2^3(t) + \mathbf{b}_3 B_3^3(t)}{B_0^3(t) + w_1 B_1^3(t) + w_2 B_2^3(t) + B_3^3(t)}, \tag{2}$$

where the $\mathbf{b}_i$ are 2D *control points* and the B_i^3 are cubic *Bernstein polynomials*. The real numbers w_1, w_2 are called weights. See [6] or [5].

An arc of a circle may be written as a rational quadratic:

$$\mathbf{x}(t) = \frac{\mathbf{c}_0 B_0^2(t) + v_1 \mathbf{c}_1 B_1^2(t) + \mathbf{c}_2 B_2^2(t)}{B_0^2(t) + v_1 B_1^2(t) + B_2^2(t)} \tag{3}$$

where the control points $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ form an isosceles triangle with base $\mathbf{c}_0, \mathbf{c}_2$. If the base angle of the triangle is α, then $v_1 = \cos\alpha$. The control point $\mathbf{c}_1$ is the intersection of the lines given by $\mathbf{c}_0, \mathbf{v}_0$ and $\mathbf{c}_2, \mathbf{v}_1$ where $\mathbf{v}_0$ and $\mathbf{v}_1$ are the unit tangent vectors at $\mathbf{c}_0$ and $\mathbf{c}_2$, resp.

We may degree elevate to obtain a rational cubic representation of the same circular arc with control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and weights $1, w_1, w_2, 1$:

$$w_1 = w_2 = \frac{1}{3}[1 + 2v_1], \tag{4}$$

$$\mathbf{b}_0 = \mathbf{c}_0, \quad \mathbf{b}_1 = \frac{\mathbf{c}_0 + 2v_1\mathbf{c}_1}{1 + 2v_1}, \quad \mathbf{b}_2 = \frac{2v_1\mathbf{c}_1 + \mathbf{c}_2}{1 + 2v_1}, \quad \mathbf{b}_3 = \mathbf{c}_2. \tag{5}$$

We now refer to Fig. 3. Let $l_0 = \|\mathbf{c}_1 - \mathbf{c}_0\|$ and $r_0 = \|\mathbf{b}_1 - \mathbf{b}_0\|$, also $l = \|\mathbf{b}_3 - \mathbf{b}_0\|/2$. Then

$$l_0 = \frac{l}{\cos\alpha}$$

and

$$r_0 = \frac{2v_1}{1 + 2v_1} l_0$$

and thus

$$r_0 = \frac{2l}{1 + 2v_1}. \tag{6}$$

This implies

$$\mathbf{b}_1 = \mathbf{b}_0 + \frac{2l}{1 + 2\cos\alpha}\mathbf{v}_0 \qquad w_1 = \frac{1}{3}[1 + 2\cos\alpha] \tag{7}$$

Note that we have thus been able to find $\mathbf{b}_1$ without having to consider $\mathbf{c}_1$. This will be advantageous in Section 6.

In the same manner, we can find $\mathbf{b}_2$ and w_2.

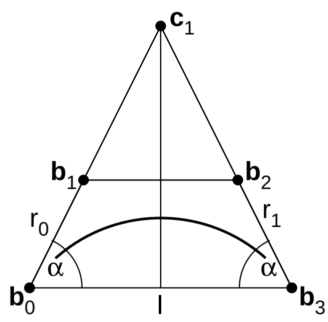

Figure 3: A circle in rational quadratic and rational cubic form.

5 Rational Hermite Interpolation

The Hermite interpolation problem concerns interpolation to the two endpoints of a curve segment and two derivative vectors there. For a rational cubic $\mathbf{x}(t)$, the endpoints are $\mathbf{b}_0$ and $\mathbf{b}_3$. The derivative vectors are $\dot{\mathbf{x}}(0)$ and $\dot{\mathbf{x}}(1)$:

$$\dot{\mathbf{x}}(0) = 3w_1[\mathbf{b}_1 - \mathbf{b}_0]; \quad \dot{\mathbf{x}}(1) = 3w_2[\mathbf{b}_3 - \mathbf{b}_2], \tag{8}$$

and this may be rewritten as

$$\dot{\mathbf{x}}(0) = 3w_1 c_0 \mathbf{v}_0; \quad \dot{\mathbf{x}}(1) = 3w_2 c_1 \mathbf{v}_1$$

with $\mathbf{v}_0 = \dot{\mathbf{x}}(0)/\|\dot{\mathbf{x}}(0)\|$ and $\mathbf{v}_1 = \dot{\mathbf{x}}(1)/\|\dot{\mathbf{x}}(1)\|$.

We have solved the Hermite interpolation problem if we can find control points $\mathbf{b}_1, \mathbf{b}_2$ and corresponding weights w_1, w_2.

We know

$$\mathbf{b}_1 = \mathbf{b}_0 + c_0\mathbf{v}_0 \quad \text{and} \quad \mathbf{b}_2 = \mathbf{b}_3 - c_1\mathbf{v}_1$$

If our rational cubic actually was obtained from degree elevation of a rational quadratic, we would have

$$w_1 = \frac{1}{3}[1 + 2\cos\alpha_0], \quad w_2 = \frac{1}{3}[1 + 2\cos\alpha_1]. \tag{9}$$

where

$$\alpha_0 = \angle(\dot{\mathbf{x}}(0), \mathbf{b}_3 - \mathbf{b}_0), \quad \alpha_1 = \angle(\dot{\mathbf{x}}(1), \mathbf{b}_3 - \mathbf{b}_0).$$

With w_1, w_2 thus fixed, we find

$$c_0 = \frac{\|\dot{\mathbf{x}}(0)\|}{3w_1} \quad \text{and} \quad c_1 = \frac{\|\dot{\mathbf{x}}(1)\|}{3w_2}.$$

This will ensure circular precision if the input data allow for a circular arc and will generate a "reasonable" curve otherwise. For the special case $\alpha_0 = \alpha_1 = 0$, we obtain a straight line which is linearly parametrized with weights $w_1 = w_2 = 1$.

For the case of either α_0 or α_1 exceeding 90 degrees, we reset to the complement $180^{\mathrm{o}} - \alpha_i$.

6 Tangent length and weight estimation

In many cases, the tangent vectors $\dot{\mathbf{x}}(0), \dot{\mathbf{x}}(1)$ will only be known as a direction without magnitude. We then have the following interpolation problem:

Given:
1. the two curve endpoints $\mathbf{b}_0$ and $\mathbf{b}_3$,
2. the two curve end tangent directions $\mathbf{v}_0, \mathbf{v}_1$, both of unit length.

Find:
the Bézier points $\mathbf{b}_1, \mathbf{b}_2$ and the corresponding weights w_1, w_2.

Among the infinitely many solutions to this problem, we attempt to find one which is "close" to a circle. Thus, the input data permitting, the solution should be an arc of a circle.

In general, the data from 1. and 2. will not be compatible with forming a circle. In this case, the end tangents $\mathbf{v}_0$ and $\mathbf{v}_1$ will form two angles α_0 and α_1 with the base $\mathbf{b}_0, \mathbf{b}_3$. We may now apply (7) to each endpoint and obtain Bézier points

$$\mathbf{b}_1 = \mathbf{b}_0 + \frac{2l}{1 + 2\cos\alpha_0}\mathbf{v}_0. \tag{10}$$

$$\mathbf{b}_2 = \mathbf{b}_3 - \frac{2l}{1 + 2\cos\alpha_1}\mathbf{v}_1 \tag{11}$$

and weights according to (9). See Fig. 4 for an illustration using data coming from a semicircle.

Note that we do not encounter singularities for α_0 or α_1 being right angles. However, singularities arise for $1 + 2\cos\alpha_i = 0$, i.e. for $\alpha_1 = 120^o$ or $\alpha_2 = 120^o$. At the singularity, no solution exists. For angles exceeding 120^o, negative weights will arise. Subdivision will remedy this but was not pursued for this paper. Instead, a restriction on the ranges for α_0, α_1 is imposed at the end of this section.

Figure 5 shows a rational curve interpolating to two endpoints and to two tangent directions together with the corresponding curvature derivative plots. The corresponding geometric polynomial Hermite curve is shown in the two plots below. Their tangent lengths were determined by setting

$$\tau = \|\mathbf{b}_1 - \mathbf{b}_0\| = \|\mathbf{b}_2 - \mathbf{b}_3\| = 0.4 \times \|\mathbf{b}_3 - \mathbf{b}_0\|,$$

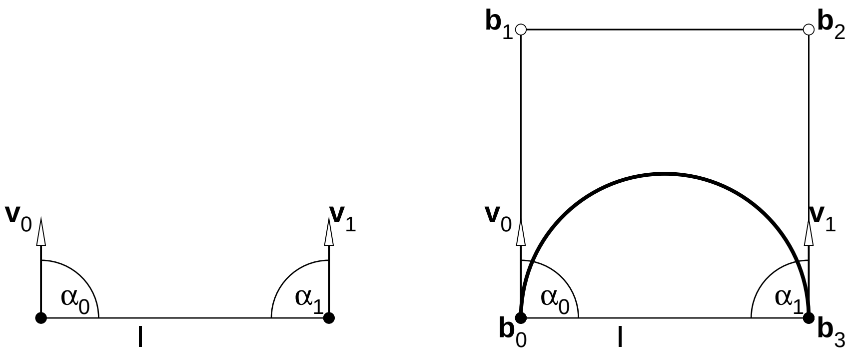

Figure 4: A rational geometric Hermite interpolant (right) to data from a semicircle (left). Weights: 1,1/3,1/3,1.

i.e., both tangents lengths were given the same value. This choice will be discussed below.

Note how our rational interpolant shows markedly less variation (1.2 variation from max to min) in curvature than the polynomial Hermite one (8 from max to min).

In order to compare our rational scheme to cubic Hermite interpolation for more than a few examples, we ran the following suite of tests. First, we note that fixing two endpoints $\mathbf{b}_0 = (-1, 0)$ and $\mathbf{b}_3 = (1, 0)$ presents no loss of generality. The two end tangents are then determined by the angles α_0 and α_1. Thus $s(\mathbf{x})$ of (1) only depends on the two parameters α_0, α_1 and will be rewritten as $s(\alpha_0, \alpha_1)$.

The quality of a (generalized) Hermite interpolation problem could thus be measured by

$$q(\mathbf{x}) = \int_a^b \int_c^d s(\alpha_0, \alpha_1) \mathrm{d}\alpha_1 \mathrm{d}\alpha_0 \tag{12}$$

where a, b, c, d are limitations of admissible values for α_0, α_1. We settled for $a = -90^o, b = 90^o$ and $c = 90^o, d = 270^o$, i.e. all admissible $\mathbf{v}_0$ have a positive $x-$ component, and all admissible $\mathbf{v}_1$ have a negative $x-$ component. For values outside this range, both Hermite and our rational Hermite could produce cusps.

We then found (using Mathematica)

$$q(\mathbf{x}) = 241.6.$$

In order to compare to geometric cubic Hermite interpolation in a fair way, we needed to find an optimal tangent length τ for geometric cubic Hermite interpolants $\mathbf{h}(t)$. This value was found to be $\tau = 0.4$ by testing (12) for a several choices of τ.[3]

[3]if α_0, α_1 were allowed to vary between $-180^o, +180^o$, then $\tau = 0.8$ gave a better value.

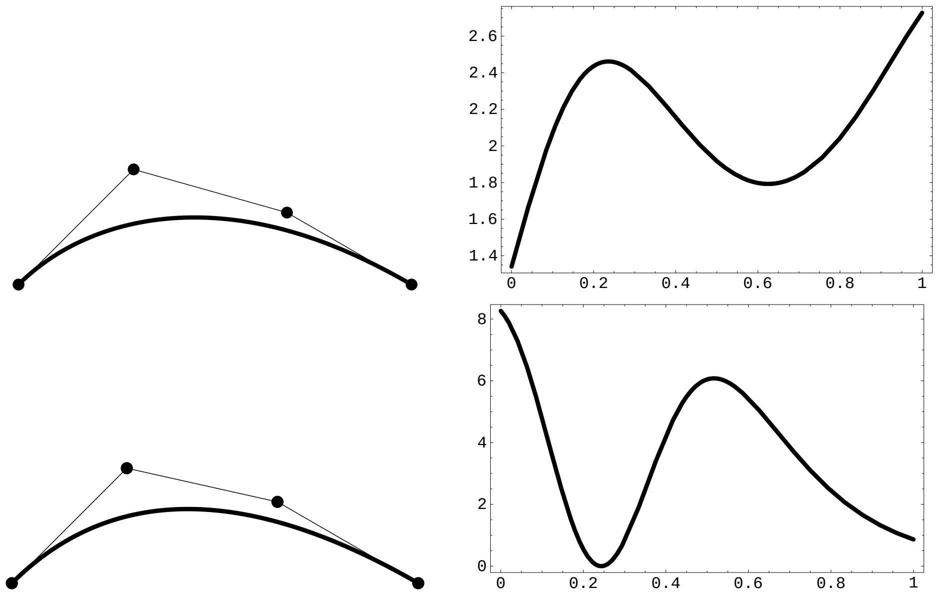

Figure 5: Top left, rational geometric Hermite interpolant. Top right: curvature derivative plot. Bottom left: polynomial Hermite interpolant. Bottom right: curvature derivative plot.

With this optimal value for τ, we found

$$q(\mathbf{h}) = 375.3,$$

thus asserting that our proposed scheme outperforms the (optimal) geometric cubic Hermite interpolant. It should be noted, however, that for about 5% of all cases, the polynomial Hermite scheme did better than ours.

7 Conclusion and Future Work

We presented several 2D interpolation schemes which all strive to produce curves close to circles. The schemes also work for 3D, but this was not tested – it may be more desirable to produce parts of helix in that context.

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